ON GROWTH RATE AND CONTACT HOMOLOGY

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Abstract. It is a conjecture of Colin and Honda that the number of Reeb periodic orbits of universally tight contact structures on hyperbolic manifolds grows exponentially with the period, and they speculate further that the growth rate of contact homology is polynomial on non-hyperbolic geometries. Along the line of the conjecture, for manifolds with a hyperbolic component that fibers on the circle, we prove that there are infinitely many non-isomorphic contact structures for which the number of Reeb periodic orbits of any non-degenerate Reeb vector field grows exponentially. Our result hinges on the exponential growth of contact homology which we derive as well. We also compute contact homology in some non-hyperbolic cases that exhibit polynomial growth, namely those of universally tight contact structures non-transverse to the fibers on a circle bundle.

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1. Introduction and main results

The goal of this paper is to study connections between the geometry of a manifold and the asymptotic number of Reeb periodic orbits with period smaller than L as $L \to \infty$. We first recall some basic definitions of contact geometry. A 1-form α on a 3-manifold M is called a *contact form* if $\alpha \wedge \mathrm{d}\alpha$ is a volume form on M. A *contact structure* ξ is a plane field locally defined as the kernel of a contact form. In what follows, we will always assume that contact structures are co-orientable. If ξ is co-oriented, it is globally defined by a (non unique) contact form. The *Reeb vector field* associated to a contact form α is the vector field R_α such that $\iota_{R_\alpha}\alpha = 1$ and $\iota_{R_\alpha}\mathrm{d}\alpha = 0$. It strongly depends on α. The Reeb vector field (or the associated contact form) is called *hypertight* if there is no contractible periodic orbit. It is called *non-degenerate* if all periodic orbits are non-degenerate (1 is not an eigenvalue of the first return map).

A fundamental step in the classification of contact structures was the definition of tight and overtwisted contact structures given by Eliashberg [20] in the line of Bennequin's work [1]. A contact structure ξ is *overtwisted* if there exists an embedded disk tangent to ξ on its boundary. Otherwise ξ is said to be *tight*. *Universally tight* contact structures are structures admitting a tight lift on universal cover. Universally tight and hypertight [29] contact structures are always tight.

To get information on the contact structure from the Reeb vector field, one usually focuses on Reeb periodic orbits. On closed 3-manifolds, Reeb vector fields always admit a periodic orbit. This is not true for a general vector field: Kuperberg [37] constructed a smooth vector field on S^3 without periodic orbits. This theorem of Taubes [45] is the 3-dimensional case of Weinstein conjecture. Beyond the existence of a single periodic Reeb orbit, Colin and Honda are interested in the number $N_L(\alpha)$ of Reeb periodic orbits with period at most L and they connect it to the Thurston geometry of the manifold.

Conjecture 1.1 (Colin-Honda [17, Conjecture 2.10])**.** *For all non-degenerate contact form α of a universally tight contact structure on hyperbolic closed 3-manifolds, $N_L(\alpha)$ exhibits an exponential growth.*

We focus on manifolds with a non-trivial JSJ decomposition including a hyperbolic component that fibers on the circle (see [2] for more information). The following theorem is one of the main results of this text.

Theorem 1.2. *Let M be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into irreducible manifolds including a hyperbolic component that fibers on the circle. Then, M carries an infinite number of non-isomorphic, hypertight, universally tight contact structures such that for all hypertight non-degenerate contact form α, $N_L(\alpha)$ grows exponentially with L.*

If contact homology is well defined and invariant[1]*, $N_L(\alpha)$ grows exponentially if α is only non-degenerate.*

The exponential growth of the number of Reeb periodic orbits for a universally tight contact structure on a hyperbolic closed manifold that fibers on the circle, a special case of Conjecture 1.1, remains an open problem. In addition, if Thurston's "virtually fibered" conjecture [46] is confirmed, hyperbolic manifolds that fibers on the circle will become the general situation up to finite covering. Very recently, Agol announced a proof of this conjecture[2].

Introduced in the vein of Floer homology by Eliashberg, Givental and Hofer in 2000 [21], *contact homology* and more generally *Symplectic Field Theory (SFT)* is an invariant of the contact structure computed through a Reeb vector field R_α. The complex is the super-commutative $\mathbb{Q}$-algebra generated by Reeb periodic orbits and the differential "counts" pseudo-holomorphic curves in the symplectisation[3] $(\mathbb{R}\times M, \mathrm{d}(e^\tau\alpha))$. Computation of contact homology hinges on finding periodic orbits and solving elliptic partial differential equations and thus is usually out of reach. The *growth rate of contact homology* is an invariant derived from contact homology introduced by Bourgeois and Colin [6]. It "describes" the asymptotic behavior of the number of Reeb periodic obits with period smaller that L that contribute to contact homology. It is the contact equivalent of the growth rate of symplectic homology introduced by Seidel [44] and used by McLean [42] to distinguish between cotangent bundles and smooth affine varieties. Theorem 1.2 is a corollary of Theorem 1.3.

Theorem 1.3. *Let M be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into irreducible manifolds including a hyperbolic component that fibers on the circle. Then, M carries an infinite number of non-isomorphic, hypertight, universally tight contact structures with an exponential growth rate of contact homology restricted to primitive classes.*

Under Hypothesis H, the growth rate of linearized contact homology is exponential.

This result draws its inspiration in Colin and Honda's results [17] on exponential growth of contact homology for contact structures adapted to an open book with pseudo-Anosov monodromy. As proved by Thurston [47], a manifold that fibers on the circle is hyperbolic if and only if it is the suspension of a surface by a diffeomorphism homotopic to a pseudo-Anosov.

Colin and Honda speculate further that the growth rate of contact homology is polynomial in non-hyperbolic situations.

Conjecture 1.4 (Colin-Honda)**.** *On manifolds with spherical geometry, the growth rate of contact homology for universally tight contact structures is linear.*

[1]Though commonly accepted, existence and invariance of contact homology remain unproved. Some results in this paper depend on these properties. In what follows this assumption will be called Hypothesis H, see section 2 for more details.

[2]see http://ldtopology.wordpress.com/2012/03/12/or-agols-theorem/

[3]τ is the $\mathbb{R}$-coordinate.

On manifolds with a geometric structure neither hyperbolic nor spherical, the growth rate of contact homology for universally tight contact structures is usually polynomial.

In this text, we study contact structures on circles bundles. Giroux [27] and Honda [33] classified them independently. Figure 1 gives a summary of this classification. Statements such as "tangent to the fibers" or "transverse to the fibers" mean that there exists an isotopic contact structure with this property, $\chi(S)$ is the Euler characteristic and $\chi(S,V)$ the Euler class.

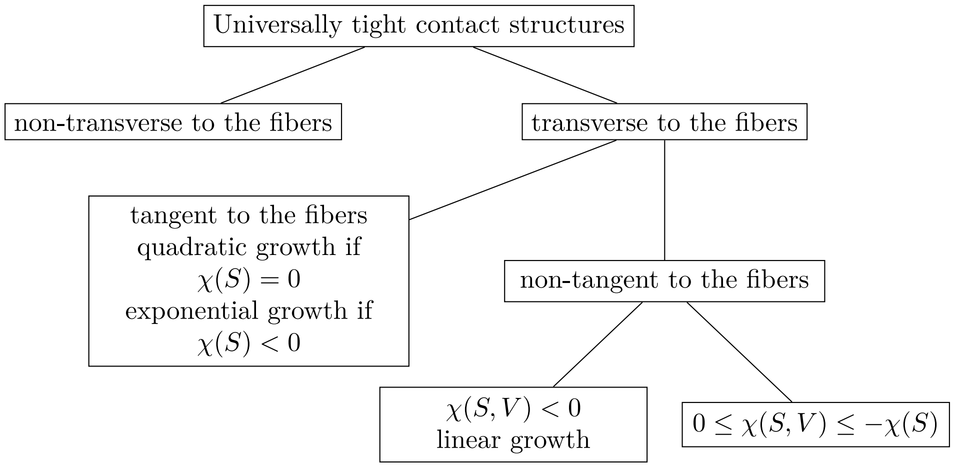

FIGURE 1. Universally tight contact structures on circle bundles over a surface S of non-positive Euler characteristic.

In some cases, the contact homology and its growth rate are already known. For instance, contact structures tangent to the fibers are fibered covering of $(UTS, \xi_{\mathrm{std}})$ where UTS is the unitary tangent bundle over S and ξ is the contact element contact structure (see [25]). In this case, the Reeb flow of the standard contact form associated to a Riemannian metric is the geodesic flow. If the surface is hyperbolic, there exists an unique closed geodesic in each homotopy class [36, Theorem 3.9.5] and the number of homotopy classes has exponential growth with respect to length [43]. Therefore, growth rates of the number of periodic Reeb orbits and of contact homology are exponential. This is an exception to the second statement of Conjecture 1.4.

If S is a torus, universally tight contact structures are standard contact structures on T^3 [26], the contact homology is known and its growth rate is quadratic (see [3]). Bourgeois [3] also studied contact structures transverse and non-tangent to the fibers with $\chi(S,V) < 0$. By use of Morse-Bott theory, he computed contact homology and obtained a linear growth rate. These structures have a S^1-invariant contact structure in their isotopy class.

In this text we study universally tight contact structures non-transverse to the fibers.

Definition 1.5 (Giroux [27])**.** A contact structure ξ on a fiber bundle $\pi : M \to S$ is *walled* by an oriented multi-curve Γ on S if

(1) ξ is transverse to the fibers on $M \setminus \pi^{-1}(\Gamma)$;
(2) ξ is transverse to $\pi^{-1}(\Gamma)$ and tangent to the fiber on $\pi^{-1}(\Gamma)$.

Walled contact structures admit a S^1-invariant walled contact structure in their isotopy class.

Theorem 1.6 (Giroux [27]). *Universally tight contact structures non-transverse to the fibers are exactly contact structures isotopic to a contact structure walled by a non-trivial multi-curve with no contractible component.*

Theorem 1.7 is the second main result of this paper.

Theorem 1.7. *Let (M, ξ) be a fiber bundle over a closed oriented surface carrying a contact structure walled by a non-trivial multi-curve $\Gamma = \bigcup_{i=0}^{n} \Gamma_i$ with no contractible component. If $X = M \setminus \pi^{-1}(\Gamma)$, let $X_1^+ \dots X_{n_+}^+$ denote its connected components for which ξ is positively transverse to the fibers and $X_1^- \dots X_{n_-}^-$ denote those for which ξ is negatively transverse to the fibers. Let a be a loop in M. Then, there exists a hypertight contact equation α such that the cylindrical contact homology $HC_*^{[a]}(M, \alpha, \mathbb{Q})$ is well defined and*

(1) *if $[a] = [fiber]^k$ and $\pm k > 0$, then $HC_*^{[a]}(M, \alpha, \mathbb{Q}) = \bigoplus_{j=1}^{n_\pm} H_*(W_j^\pm, \mathbb{Q})$;*

(2) *if $[a] = [fiber]^k [\Gamma_j]^{k'}$ and[4] $k' \neq 0$, then $HC_*^{[a]}(M, \alpha, \mathbb{Q}) = \bigoplus_{[\Gamma_i]=[\Gamma_j]} H_*(S^1, \mathbb{Q})$;*

(3) *otherwise, $HC_*^{[a]}(M, \alpha, \mathbb{Q}) = 0$.*

Under Hypothesis H, these contact homologies are the homologies $HC_*^{[a]}(M, \xi, \mathbb{Q})$ and the growth rate of contact homology is quadratic.

It remains to compute contact homology of contact structures transverse to the fibers with $0 \leq \chi(S, V) \leq -\chi(S)$.

Colin and Honda's conjectures remain out of reach as we are rather ignorant of contact structures on hyperbolic manifolds. As observed above, there is actually a counterexample to Conjecture 1.4. This suggests that we need more examples to refine the statement of this conjecture. The following questions provide some alternative way to tackle connections between geometry and Reeb periodic orbits.

Question 1. *Is the growth rate of contact homology related to that of the fundamental group ?*

Question 2. *Are there growth rates of contact homology that lie between quadratic and exponential growths ?*

This paper is derived from the PhD thesis of the author [49]. This text is organized as follows. In Section 2, we introduce contact homology, our main tool to study Reeb periodic orbits. Morse-Bott contact homology, outlined in Section 3, is a generalization of contact homology and is a significant ingredient in the proof on Theorem 1.7. In Section 4, we give a detailed definition of the growth rate of contact homology. Though this definition dates from 2005, there is no complete description and proof of invariance. Positivity of intersection helps to control holomorphic cylinders and is an important ingredient in the proofs of Theorems 1.2 and 1.7. In Section 5 we discuss positivity of intersection for tori foliated by Reeb orbits. Section 6 contains the proof of Theorem 1.7 and Section 7 the proof of Theorem 1.2.

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[4] The fiber is central in $\pi_1(V)$, thus products of a free homotopy class with the fiber are well-defined.

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2. Contact homology

Gromov [28] introduced pseudo-holomorphic curves in the symplectic world in 1985 for compact manifolds. Hofer [29] generalized them to symplectisation to study the Weinstein conjecture. Pseudo-holomorphic curves are a key ingredient in the *contact homology* introduced by Eliashberg, Givental and Hofer [21]. Here we consider contact homology over $\mathbb{Q}$. For a more general definition over a Novikov ring one can refer to [5]. A complete proof of existence and invariance of contact homology is still missing. This is mainly due to severe transversality issues for multiply-covered curves.

2.1. **Almost-complex structures and holomorphic curves.** The *symplectisation* of a contact manifold $(M,\xi=\ker(\alpha))$ is the non-compact symplectic manifold $(\mathbb{R}\times M, \mathrm{d}(e^\tau\alpha))$ where τ is the $\mathbb{R}$-coordinate. An *almost complex structure* on a even-dimensional manifold M is a map $J:TM\to TM$ preserving the fibers and such that $J^2=-\mathrm{Id}$. An almost complex structure J on a symplectisation is *adapted* to α if

(1) J is τ-invariant;
(2) $J\frac{\partial}{\partial\tau}=R_\alpha$;
(3) $J\xi=\xi$;
(4) $\omega(\cdot,J\cdot)$ is a Riemannian metric[5].

A map $u:(M_1,J_1)\to(M_2,J_2)$ is *pseudo-holomorphic* if $\mathrm{d}u\circ J_1=J_2\circ\mathrm{d}u$. One can refer to [41] for more information.

Theorem 2.1 ([41, Lemma 2.4.1]). *Let U be an open subset of a Riemann surface (S,j) and (M,J) be an manifold with an almost complex structure. Then, the critical points of any non-constant J-pseudo-holomorphic map $u:(U,j)\to(M,J)$ are isolated.*

To define contact homology, we consider pseudo-holomorphic maps $u:(\dot\Sigma,j)\to\mathbb{R}\times M$ where $(\dot\Sigma,j)$ is a punctured Riemannian surface. The simplest non-constant holomorphic maps are trivial cylinders: if γ is a T-periodic Reeb orbit, the associated *trivial cylinder* is

$$\begin{array}{ccc}\mathbb{R}\times S^1 & \longrightarrow & \mathbb{R}\times M\\ (s,t) & \longmapsto & (Ts,\gamma(Tt)).\end{array}$$

Recall that the *Hofer energy* E of $u=(a,f):\dot\Sigma\to\mathbb{R}\times V$ is

$$E(u)=\sup\left\{\int_{\dot\Sigma}u^*\mathrm{d}(\varphi\alpha),\varphi:\mathbb{R}\to[0,1],\varphi'\geq 0\right\}$$

If $u=(a,f)$ is a pseudo-holomorphic map, then $E(u)\geq 0$. If $E(u)=0$, the image of f is contained in a Reeb trajectory (see [8, 5.3]). Choose some polar coordinates (ρ,θ) centered on the puncture x of $\dot\Sigma$ in a neighborhood of x. The map $u=(a,f):\dot\Sigma\to\mathbb{R}\times M$ is *positively asymptotic* to a T-periodic orbit γ in a neighborhood of x if

(1) $\lim_{\rho\to 0}a(\rho,\theta)=+\infty$;
(2) $\lim_{\rho\to 0}f(\rho,\theta)=\gamma(-T\theta)$.

It is *negatively asymptotic* to γ if

(1) $\lim_{\rho\to 0}a(\rho,\theta)=-\infty$;

[5]On a symplectic manifold (M,ω), an almost complex structure is *compatible* if $\omega(\cdot,J\cdot)$ is a Riemannian metric.

(2) $\lim_{\rho\to 0} f(\rho,\theta) = \gamma(+T\theta)$.

We now assume that α is non-degenerate. It is a theorem of Hofer [29, Theorem 31] that holomorphic curves $u : (\dot\Sigma, j) \to (R\times M, J)$ with finite Hofer energy are asymptotic to a Reeb periodic orbit near each puncture. By Stokes theorem, if $u : \dot\Sigma \to \mathbb{R}\times M$ is a holomorphic map with finite Hofer energy, positively asymptotic to $\gamma_1^+, \dots, \gamma_{n_+}^+$ and negatively asymptotic to $\gamma_1^-, \dots, \gamma_{n_-}^-$. Then

$$E(u) = \sum_{i=1}^{n_+} \mathcal{A}(\gamma_i^+)$$

where $\mathcal{A}(\gamma) = \int_\gamma \alpha$ is the period of γ and represents the action

The following proposition is used in Section 5 to prove smoothness of the projection of a holomorphic curve on M.

Proposition 2.2 (see [41, Lemma 2.4.1])**.** *Let (M,ξ) be a contact manifold, α a contact equation and J an almost complex structure on $(\mathbb{R}\times M, \mathrm{d}(e^\tau\alpha))$. Consider the standard complex structure j on $\mathbb{R}\times S^1$. For every non-constant map $u : (\mathbb{R}\times S^1, j) \to (R\times M, J)$ which is not a trivial cylinder, the points (s,t) such that $\frac{\partial}{\partial\tau} \in \mathrm{im}(\mathrm{d}u(s,t))$ are isolated.*

2.2. **Full contact homology.** Consider $(V, \xi = \ker(\alpha))$ a contact manifold, γ a T-periodic Reeb orbit and $p\in\gamma$. Let φ_t denote the Reeb flow. The map $\mathrm{d}\varphi_T(p) : (\xi_p, \mathrm{d}\alpha) \to (\xi_p, \mathrm{d}\alpha)$ is a symplectomorpism. A non-degenerate periodic orbit γ is called *even* if $\mathrm{d}\varphi_T(p)$ has two real positive eigenvalues and *odd* if $\mathrm{d}\varphi_T(p)$ has two complex conjugate or two real negative eigenvalues. Let γ_m be the m-th multiple of a simple orbit γ_1. Then γ_m is said to be *good* if γ_1 and γ_m have the same parity, otherwise, γ_m is said to be *bad*. The *Conley-Zehnder* index gives a relative grading of Reeb periodic orbits. Its parity matches with the above definitions.

2.2.1. *Definition.* Let $(M, \xi = \ker(\alpha))$ be a contact manifold with a non-degenerate contact equation. We sketch the construction of full contact homology chain complex $(A_*(M,\alpha), \partial)$ defined in [21] by Eliashberg, Givental and Hofer. The chain complex $A_*(M,\alpha)$ is the super-commutative $\mathbb{Q}$-algebra generated by good Reeb periodic orbits (here we consider simple periodic orbits and their good multiples). Choose an almost complex structure J adapted to the symplectisation. To define $\partial\gamma$, consider the set

$$\mathcal{M}_{[Z]}(J, \gamma, \gamma_1', \dots \gamma_n').$$

of equivalence classes (modulo reparametrization of $\dot\Sigma$) of solutions of the Cauchy-Riemann equation with finite energy, positively asymptotic to γ, negatively asymptotic to $\gamma_1' \dots \gamma_n'$ and in the relative homotopy class $[Z]$. The $\mathbb{R}$-translation in $\mathbb{R}\times M$ induce a $\mathbb{R}$-action on $\mathcal{M}_{[Z]}(J, \gamma, \gamma_1', \dots \gamma_n')$ (see [5] for more details).

Hypothesis H. *There exists an abstract perturbation of the Cauchy-Riemann equation such that*

$$\overline{\mathcal{M}}_{[Z]}(J, \gamma, \gamma_1', \dots \gamma_n') = \mathcal{M}_{[Z]}(J, \gamma, \gamma_1', \dots \gamma_n')/_{\mathbb{R}}$$

is a union of branched labeled manifolds with corners and rational weights whose dimensions are given by $[Z]$ and the Conley-Zehnder indices of asymptotic periodic orbits.

Fredholm theory for multi-covered curves is not written anywhere. There exists different approches to the perturbation of moduli spaces due to Fukaya and Ono [24], Liu and Tian [39], Hofer, Wysocki and Zehnder [30, 32, 31] (see [32]) or Cieliebak and Oancea in the equivariant contact homology setting [13] (see [10]). There also exist partial transversality results due to Dragnev [19].

Let $n_{\gamma,\gamma'_1,\dots\gamma'_n}$ denote the signed weighted counts of points in 0-dimensional components of $\overline{\mathcal{M}}_{[Z]}(J,\gamma,\gamma'_1,\dots\gamma'_n)$ [21, 9] for all relative homology classes $[Z]$. The differential of a periodic orbit γ is

$$\partial\gamma = \sum_{\gamma'_1\cdots\gamma'_n} \frac{n_{\gamma,\gamma'_1,\dots\gamma'_n}}{i_1!\dots i_l!\kappa(\gamma'_1)\dots\kappa(\gamma'_n)}\gamma'_1\dots\gamma'_n$$

where $i_1\dots i_l$ are multiplicities in $\{\gamma'_1\dots\gamma'_n\}$ and $\kappa(\gamma)$ is the multiplicity of γ. The definition is extended using the graded Leibniz rule.

Under Hypothesis H, it is reasonable to expect the following: if there exists an open set $U\subset\mathbb{R}\times M$ containing all the images of J-holomorphic curves positively asymptotic to γ, negatively asymptotic to $\gamma'_1\dots\gamma'_n$, then U contains the images of all solutions of perturbed Cauchy-Riemann equations with the same asymptotics for all small enough abstract perturbations.

The Hypothesis H is the key ingredient to get existence and invariance of contact homology.

Theorem 2.3 (Eliashberg-Givental-Hofer)**.** *Under Hypothesis H,*

(1) $\partial^2=0$*;*
(2) *the associated homology $HC_*(M,\xi)$ does not depend on the choice of the contact form, complex structure and abstract perturbation.*

If $\partial^2=0$ for some contact form α, we denote $HC_*(M,\alpha,J)$ the associated homology.

Some computations were carried out by Bourgeois and Colin [6] to distinguish toroidal irreducible 3-manifolds, Ustilovsky [48] to prove the existence of exotic contact structures on spheres and Yau [50] who proved that the contact homology of overtwisted contact structures is trivial. Bourgeois [3] provided other computations using Morse-Bott contact homology.

2.2.2. *Changing contact form.* The proof of invariance of contact homology hinges on constructing maps between chain complexes associated to different contact forms. One can refer to [3] or [17] for more details. These maps are useful to prove invariance of the growth rate of contact homology.

First, we consider proportional contact forms. Let α be a non-degenerate contact form of (M,ξ) and J be an adapted almost complex structure. For all $c>0$, consider the adapted almost complex structure J^c such that $J^c_{|\xi}=J_{|\xi}$ and $J^c\frac{\partial}{\partial\tau}=\frac{R_\alpha}{c}$. The diffeomorphism

$$\begin{array}{rccc}\varphi_c: & \mathbb{R}\times V & \longrightarrow & \mathbb{R}\times V\\ & (\tau,x) & \longmapsto & (c\tau,x)\end{array}$$

sends a J-holomorphic curve on a J^c-holomorphic curve. The identification of geometric Reeb periodic orbits induce an isomorphism $\theta(\alpha,J,c)$ between the chain complexes $(A_*(M,\alpha),\partial_J)$ and $(A_*(M,c\alpha),\partial_{J^c})$. Let $\Theta(\alpha,J,c)$ denote the induced map on homology.

Let α_1 and α_0 be two non-degenerate, homotopic contact forms. Then there exist $c>0$ and a homotopy $(\alpha_t)_{t\in\mathbb{R}}$ such that

(1) $\lim\limits_{t\to-\infty}\alpha_t=c\alpha_0$;
(2) $\lim\limits_{t\to\infty}\alpha_t=\alpha_1$;
(3) if α denote the induced form on $\mathbb{R}\times M$, then $\mathrm{d}\alpha\wedge\mathrm{d}\alpha>0$.

Choose an almost complex structure J compatible with $\mathrm{d}\alpha$ and interpolating between two almost complex structures J_1 and J_0^c adapted to α_1 and $c\alpha_0$. There exists a chain map counting J-holomorphic curves

$$\psi((\alpha_1,J_1),(c\alpha_0,J_0^c)):(A_*(M,\alpha_1),\partial_{J_1})\to(A_*(M,c\alpha_0),\partial_{J_0^c}).$$

This map decreases the action by Stokes' theorem. The induced map in homology

$$\Psi((\alpha_1, J_1), (c\alpha_0, J_0^c)) : HC_*(M, \alpha_1, J_1), \to HC_*(M, c\alpha_0, J_0^c).$$

does not depend on α_t or J. These maps have natural composition properties as stated in the following theorem.

Theorem 2.4 (Eliashberg-Givental-Hofer). *On a closed* 3*-manifold, under Hypothesis H,*

(1) *if* $\Psi((\alpha_2, J_2), (\alpha_1, J_1))$ *and* $\Psi((\alpha_1, J_1), (\alpha_0, J_0))$ *are well-defined, then*

$$\Psi((\alpha_2, J_2), (\alpha_0, J_0)) = \Psi((\alpha_1, J_1), (\alpha_0, J_0)) \circ \Psi((\alpha_2, J_2), (\alpha_1, J_1)).$$

(2) $\Psi((\alpha_1, J_1), (\alpha_1, J_1'))$ *and* $\Psi((\alpha_1, J_1'), (\alpha_1, J_1))$ *always exist.*
(3) *For all* $c > 0$,

$$\Theta(\alpha_0, J_0, c) \circ \Psi((\alpha_1, J_1), (\alpha_0, J_0)) = \Psi((c\alpha_1, J_1^c), (c\alpha_0, J_0^c)) \circ \Theta(\alpha_1, J_1, c).$$

(4) *if* $c < 1$, *one can choose* $\psi((\alpha, J), (c\alpha, J^c)) = \theta(\alpha, J, c)$.

Sketch of proof. (3) Denote α_t and J the homotopy and almost complex structure used to define $\psi((\alpha_1, J_1), (\alpha_0, J_0))$. Consider the homotopy $c\alpha_{\frac{t}{c}}$ and the almost complex structure $J^c = \varphi_* J$ where $\varphi : (\tau, x) \mapsto (c\tau, x)$. Then φ sends J-holomorphic curves to J^c-holomorphic curves.

(4) Consider the homotopy $\alpha_t = c(t)\alpha_0$ between $c\alpha_0$ and α_0 where $t \longmapsto c(t)$ is a non-decreasing function. Let J_0 be an almost complex structure adapted to α_0 and C be an antiderivative of c. The almost complex structure $J = \varphi_* J_0$ where $\varphi : (\tau, x) \longmapsto (C(\tau), x)$ is adapted to α_t. Then φ sends J_0-holomorphic curves to J-holomorphic-curves and the J_0-holomorphic curves used to define

$$\psi((\alpha_0, J_0), (\alpha_0, J_0))$$

are trivial cylinders. □

2.3. **Cylindrical contact homology.** Let (M, ξ) be a closed contact manifold and α be a non-degenerate hypertight contact form. The chain complex $(C_*^{\mathrm{cyl}}(M, \alpha), \partial)$ of *cylindrical contact homology* is the $\mathbb{Q}$-vector space generated by good Reeb periodic orbits associated to the form α. Choose an almost complex structure J adapted to the symplectisation. The differential of a periodic orbit γ is

$$\partial\gamma = \sum_{\gamma'} \frac{n_{\gamma,\gamma'}}{\kappa(\gamma')}\gamma'.$$

As in full contact homology situation, the case of multiply covered cylinders is knotty.

Theorem 2.5 (Eliashberg-Givental-Hofer). *Under Hypothesis H,*

(1) $\partial^2 = 0$;
(2) *the associated homology* $HC_*^{cyl}(M, \xi)$ *does not depend on the choice of a contact form* α*, an almost complex structure* J *and an abstract perturbation.*

Nevertheless there exists a well defined and invariant partial version of cylindrical contact homology.

Definition 2.6. Let Λ be a set of free homotopy classes of M. The *partial cylindrical homology* restricted to Λ is the homology of the chain complex $(C_*^\Lambda(M, \alpha), \partial)$ where $C_*^\Lambda(M, \alpha)$ is generated by good Reeb periodic orbits in Λ and ∂ is the restriction of the cylindrical contact homology differential.

If Λ contains only primitive free homotopy classes, Dragnev's work [19, Corollary 1] shows that for a generic almost complex structure, the partial contact homology $HC_*^{\Lambda}(M,\alpha,J)$ is well defined and does not depend on the choice of J or of a hypertight non-degenerate form: if an orbit is in a primitive homotopy class, any holomorphic cylinder asymptotic to it is somewhere injective.

Fact 2.7. *The morphisms from Theorem 2.4 induce morphisms ψ^{cyl} et Ψ^{cyl} on cylindrical contact homology complex and on cylindrical contact homology with similar properties.*

2.4. **Linearized contact homology.** Cylindrical contact homology is a special case of linearized contact homology. Introduced in Chekanov's work on Legendrian contact homology [12], linearized contact homology was generalized to contact homology by Bourgeois, Eliashberg and Ekholm [7]. One can also refer to [17].

Definition 2.8. An *augmentation* $\varepsilon : (A,\partial) \to (\mathbb{Q},0)$ is a $\mathbb{Q}$-algebra homomorphism that is also a chain map.

An augmentation ε in (A,∂) gives a "change of coordinates" $a \mapsto \overline{a} = a - \varepsilon(a)$. Let $(A^{\varepsilon}(M,\alpha),\partial^{\varepsilon})$ denote the new chain complex and write $\partial^{\varepsilon} = \partial_1^{\varepsilon} + \partial_2^{\varepsilon} + \dots$ using the filtration by word length. In particular $\partial_0^{\varepsilon} = 0$.

Proposition 2.9 (Bourgeois-Ekholm-Eliashberg)**.** *If ε is an augmentation, then $(\partial_1^{\varepsilon})^2 = 0$.*

Definition 2.10. Let $(M,\xi = \ker(\alpha))$ be a contact manifold with a non-degenerate contact form and ε be an augmentation of $A_*(M,\alpha)$. The *linearized contact homology $HC^{\varepsilon}(M,\alpha,J)$* with respect to ε is the homology of $(A_*^{\varepsilon}(M,\alpha),\partial_1^{\varepsilon})$ where $A_*^{\varepsilon}(M,\alpha)$ is the $\mathbb{Q}$-vector space generated by $\{\overline{\gamma}, \gamma \text{ good period orbit}\}$.

Proposition 2.11. *Under Hypothesis H, if the contact form α is hypertight, the complex $A_*(M,\alpha)$ admits the trivial augmentation. The linearized contact homology is then the cylindrical contact homology.*

Let α_0 and α_1 be two non-degenerate, homotopic contact forms and

$$\varphi : (A(M,\alpha_1),\partial_{J_1}) \to (A(M,\alpha_0),\partial_{J_0})$$

be a chain map. If ε_0 is an augmentation on $(A(M,\alpha_0),\partial_{J_0})$ then φ induces a *pull back augmentation* $\varepsilon_1 = \varepsilon_0 \circ \varphi$ on $(A(M,\alpha_1),\partial_{J_1})$. The morphisms ψ et Ψ described in Theorem 2.4 induce morphisms ψ^{ε_0} and Ψ^{ε_0}. We define θ^{ε_0} and Θ^{ε_0} in the same line.

Theorem 2.12 (Bourgeois-Ekholm-Eliashberg, see [17, Theorem 3.2])**.** *Under Hypothesis H,*

(1) *The set of linearized contact homologies*

$$\{HC^{\varepsilon}(M,\alpha,J), \varepsilon \text{ augmentation of } (A_*(M,\alpha),\partial_J)\}$$

is an invariant of the isotopy class of the contact structure $\xi = \ker(\alpha)$.

(2) *Let $\varphi_1,\varphi_2 : (A(M,\alpha_1),\partial_{J_1}) \to (A(M,\alpha_0),\partial_{J_0})$ be two homotopic chain maps and ε_0 be an augmentation on $(A(M,\alpha_0),\partial_{J_0})$. Let ε_1 and ε_2 denote the pull-back augmentations by φ_1 et φ_2. Then, the map*

$$\begin{array}{cccc} \varphi(\varepsilon_1,\varepsilon_2): & (A_*^{\varepsilon_1}(M,\alpha_1),\partial_{J_1}) & \longrightarrow & (A_*^{\varepsilon_2}(M,\alpha_1),\partial_{J_1}) \\ & \gamma - \varepsilon_1(\gamma) & \longmapsto & \gamma - \varepsilon_2(\gamma) \end{array}$$

induces an isomorphism $\Phi(\varepsilon_1,\varepsilon_2)$ in homology such that the diagram

$$\begin{tikzcd} HC_*^{\varepsilon_1}(M,\alpha_1,J_1) \arrow[rr, "\Phi(\varepsilon_1,\varepsilon_2)"] \arrow[dr, "\Phi_1"'] & & HC_*^{\varepsilon_2}(M,\alpha_1,J_1) \arrow[dl, "\Phi_2"] \\ & HC_*^{\varepsilon_0}(M,\alpha_0,J_0) & \end{tikzcd}$$

commutes if Φ_1 and Φ_2 are the morphisms induced by φ_1 and φ_2.

Augmentations ε_1 et ε_2 are said to be *homotopic*, see [17, 3.2] for a general definition.

3. Morse-Bott contact homology

Bourgeois introduced Morse-Bott contact homology in his PhD thesis [3] in 2002. Morse-Bott contact homology gives a way to compute contact homology when the contact form is degenerate and there exists submanifolds foliated by Reeb periodic orbits. The main idea is to compare the Morse-Bott degenerate situation to non-degenerate situations obtained by perturbing the degenerate form using a Morse function. In this text, we will only use part of the theory on simple examples to compute the contact homology of circle bundles.

3.1. **Morse-Bott perturbations.** Let $(M, \xi = \ker(\alpha))$ be a contact manifold with a contact form α and φ_t be the Reeb flow.

Definition 3.1. The form α is of *Morse-Bott type* if

(1) $\sigma(\alpha) = \{\mathcal{A}(\gamma), \gamma \text{ periodic orbit}\}$ is discrete;
(2) if $L \in \sigma(\alpha)$, then $N_L = \{p \in V, \varphi_L(p) = p\}$ is a smooth closed submanifold;
(3) the rank of $\mathrm{d}\alpha_{N_L}$ is locally constant and $T_pN_L = \ker(\varphi_L - I)$.

For instance, the standard contact form $\alpha_n = \sin(nx)\mathrm{d}y + \cos(nx)\mathrm{d}z$ on T^3 is of Morse-Bott type. The Reeb vector field is

$$R_{\alpha_n} = \begin{pmatrix} 0 \\ \sin(nx) \\ \cos(nx) \end{pmatrix}$$

and its flow preserves all tori $\{x = \text{cst}\}$. A torus $\{x = x_0\}$ is foliated by Reeb periodic orbits if and only if $\sin(nx_0)$ and $\cos(nx_0)$ are rationally dependent. Another important example is the case of a contact structure transverse to the fibers on a circle bundle and S^1-invariant: such a contact structure admits a contact form whose Reeb vector field is tangent to the fibers. The whole manifold is then foliated by Reeb periodic orbits of the same period.

The Reeb flow induces an S^1-action on N_L for all $L \in \sigma(\alpha)$. In general, the quotient space S_L is an orbifold. However in the examples studied in this text, spaces S_L will be smooth manifolds. Hence, we assume that S_L is smooth.

We now describe how to perturb a contact form α of Morse-Bott type. Fix $L \in \sigma(\alpha)$. For all $L' \in \sigma(\alpha) \cap [0, L]$, choose a Morse function $f_{L'}$ on $S_{L'}$ and extend it to $N_{L'}$ so that $\mathrm{d}f_{L'}(R_\alpha) = 0$. Then, extend it to M in such a way that its domain is contained in a small neighborhood of $N_{L'}$. Let $\overline{f}_L$ denote the sum of all these functions. Perturb the contact equation into

$$\alpha_{\lambda,L} = (1 + \lambda \overline{f}_L)\alpha.$$

Proposition 3.2 ([3]). *For all $L > 0$, there exists $\Lambda > 0$ such that for all $0 < \lambda \leq \Lambda$, the periodic orbits of $\alpha_{\lambda,L}$ with period smaller than L correspond to critical points of $f_{L'}$ on $S_{L'}$ for $L' \in \sigma(\alpha) \cap [0, L]$. Additionally, these periodic orbits are non-degenerate.*

Remark 3.3. An S^1-invariant contact structure J on the symplectization $\mathbb{R} \times M$ induces a Riemannian metric $\mathrm{d}\alpha(\cdot, J\cdot)$ on S_L.

3.2. **Morse-Bott contact homology.** Roughly speaking, the complex of Morse-Bott contact homology is generated by critical points of the functions f_L, and the differential counts generalized holomorphic cylinders. Generalized holomorphic cylinders are a combination of holomorphic curves asymptotic to periodic orbits in the spaces N_L and gradient lines in the spaces S_L. See [3] for more details, [4] for a summary of [3], or [5] for a general presentation.

Consider a family of S^1-invariant almost complex structures J_λ adapted to $\alpha_{\lambda,L}$. Generalized holomorphic cylinders are limits of J_λ-holomorphic curves when $\lambda \to 0$ and derive from two main phenomenas. On one side, holomorphic buildings appear similarly to the non-degenerate situation: up to reparametrization, a sequence converges in $\mathcal{C}^\infty$-loc to a holomorphic curve with asymptotic periodic orbits in some intermediate spaces N_L. On the other hand, when the asymptotics of two adjacent levels in a holomorphic building differ, projections on S_L grow nearer to a gradient trajectory of f_L: up to reparametrization, a sequence converges in $\mathcal{C}^\infty$-loc to a trivial cylinder over any point of the gradient trajectory. The associated compacity theorem derives from Bourgeois' thesis [3, Chapters 3 and 4]. One can also refer to [8]. In our simpler setting, Bourgeois' results lead to the following theorems.

Theorem 3.4 (Bourgeois [3])**.** *Let $\pi : M \to S$ be a circle bundle over a closed oriented surface carrying a S^1-invariant contact form α transverse to the fibers. Fix $L > 0$ and a Morse-Bott perturbation f_L induced by a Morse function $f : S \to \mathbb{R}$. Let J_λ be a family of S^1-invariant almost complex structures on $\mathbb{R} \times M$ adapted to $\alpha_{\lambda,L}$ and converging to an almost complex structure J adapted to α. Assume that (f, g) is a Morse-Smale pair where g is the Riemannian metric on S induced by J and α.*

Fix two critical points x_+ et x_- of f so that $\mathrm{index}(x_+) - \mathrm{index}(x_-) = 1$ *and let γ_+ and γ_- denote the associated Reeb periodic orbits. Then, for all small enough λ, the moduli space $\overline{\mathcal{M}}(\gamma_+, \gamma_-, J_\lambda)$ is a 0-dimensional manifold. Additionally, $\overline{\mathcal{M}}(\gamma_+, \gamma_-, J_\lambda)$ identifies with the set of gradient trajectories from x_+ to x_-, the holomorphic curves are arbitrarily close to cylinders over the gradient trajectories and the orientations induced by contact homology and Morse theory are the same.*

Theorem 3.5 (Bourgeois [3])**.** *Consider the standard contact form $\alpha = \sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$ on T^3. Fix $L > 0$ and Morse-Bott perturbation f_L induced by a Morse function $f : S^1 \to \mathbb{R}$ with two critical points. Let J_λ be a family of almost complex structures on $\mathbb{R} \times M$ adapted to $\alpha_{\lambda,L}$, S^1-invariant on $N_{L'}$ for all $L' \leq L$ and converging to an almost complex structure J.*

Fix $L' \leq L$ and let T be a torus in $N_{L'}$. Let γ_+ and γ_- denote the two periodic orbits in T associated to the critical points of f. Then for all small enough λ, the moduli space $\overline{\mathcal{M}}(\gamma_+, \gamma_-, J_\lambda)$ has exactly two elements with opposite orientations and the homomorphic curves are arbitrarily close to cylinders over gradient trajectories of f. In addition, if γ_+ and γ_- are not in the same torus, $\overline{\mathcal{M}}(\gamma_+, \gamma_-, J_\lambda)$ is empty.

Remark 3.6. This theorem generalizes to contact forms $\sin(nx)\mathrm{d}y + \cos(nx)\mathrm{d}z$ and $f(x)\mathrm{d}y + g(x)\mathrm{d}z$ if f and g have the same variations as $x \mapsto \sin(nx)$ and $x \mapsto \cos(nx)$.

These theorems derive from Bourgeois' work and do not depend on Hypothesis H. The solutions of the Cauchy-Riemann equations is the 0-set of a Fredholm section in a Banach bundle (described in [3, 5.1.1]) and thus a 3-manifold. To achieve transversality of this section, Bourgeois proves that the linearized Cauchy-Riemann

operator is surjective on its 0-set by studying its surjectivity for curves close to holomorphic curves (the curves are defined in [3, 5.3.2], the surjectivity is proved in [3, Proposition 4.13 and 5.14]) and then using an implicit function theorem [3, Proposition 5.16]. To obtain the desired moduli space, we quotient the space of solutions by the biholomorphisms of $\mathbb{R} \times S^1$ and the $\mathbb{R}$-action. The orientation issues are studied in [3, Proposition 7.6].

Corollary 3.7 (Bourgeois [3])**.** *Let M be an oriented circle bundle over a closed oriented surface S carrying a S^1-invariant contact structure ξ which is transverse to the fibers. Let f denote the homotopy class of the fiber. Then, for all $k > 0$, there exists a contact form α such that*

$$HC_*^{f^k}(M, \alpha, \mathbb{Q}) = H_*(S, \mathbb{Q}).$$

The cylindrical contact homology is trivial in all other homotopy classes. Under Hypothesis H, the growth rate of contact homology is linear.

Corollary 3.8. *Fix $n \in \mathbb{N}^*$. Let $\alpha_n = \sin(nx)\mathrm{d}y + \cos(nx)\mathrm{d}z$ be the standard contact form on T^3. Let c_y and c_z denote the free homotopy classes associated to $S^1 \times \{0\} \times S^1$ and $S^1 \times S^1 \times \{0\}$. Let $a = c_y^{n_y} c_z^{n_z}$ be a non-trivial homotopy class. Then there exists a contact form α'_n such that*

$$HC_*^a(T^3, \alpha'_n, \mathbb{Q}) = \bigoplus_{i=1}^{n} \mathbb{H}_*(S^1, \mathbb{Q}).$$

The cylindrical contact homology is trivial in all other homotopy classes. Under Hypothesis H, the growth rate of contact homology is quadratic.

Note that contact homology distinguishes between the contact structures $\xi_n = \ker(\alpha_n)$.

4. Growth rate of contact homology

4.1. **Algebraic setting.** In this text, we mainly focus on the dichotomy between polynomial and exponential growth rate.

Definition 4.1. The growth rate of $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *polynomial of order* $\leq n$ if there exists $a > 0$ such that $f(x) \leq ax^n$ for all $x \in \mathbb{R}_+$. It is *exponential* if there exist $a > 0$ and $b > 0$ such that $f(x) \geq a \exp(bx)$ for all $x \in \mathbb{R}_+$.

The *growth rate* $\Gamma(f)$ of a non-decreasing function f may itself be defined as its equivalence class under the relation : two non-decreasing functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ are *equivalent* is there exists $C > 0$ such that

$$h\left(\frac{x}{C}\right) \leq g(x) \leq h(Cx)$$

for all $x \in \mathbb{R}_+$ (see for instance [18]). This definition is more precise than the *polynomial growth rate* of f given by

$$\limsup_{x \to \infty} \frac{\log\big(\max(f(x), 1)\big)}{\log(x)}.$$

McLean [42] defines the growth rate of symplectic homology using this formula and this growth rate is common in topology, see for instance [35].

The following algebraic preliminaries are similar to [42]. A *filtered directive system* is a family of vector spaces $(E_x)_{x \in [0,\infty[}$ such that for all $x_1 \leq x_2$, there exists a linear map $\varphi_{x_1,x_2} : E_{x_1} \longrightarrow E_{x_2}$ such that

(1) $\varphi_{x_1,x_1} = \mathrm{Id}$ for all $x_1 \geq 0$
(2) $\varphi_{x_1,x_3} = \varphi_{x_2,x_3} \circ \varphi_{x_1,x_2}$ for all $0 \leq x_1 \leq x_2 \leq x_3$.

A filtered directive system admits a direct limit $E = \lim\limits_{x\to\infty} E_x$. By definition, there exists $\varphi_x : E_x \to E$ such that the following diagram commutes for all $0 \leq x_1 \leq x_2$.

$$\begin{tikzcd}
E_{x_1} \arrow[rr, "\varphi_{x_1,x_2}"] \arrow[dr, "\varphi_{x_1}"'] & & E_{x_2} \arrow[dl, "\varphi_{x_2}"] \\
& E &
\end{tikzcd}$$

In what follows, we will assume that E_x is a finite dimensional space for all $x \geq 0$.

Definition 4.2. The *growth rate* $\Gamma((E_x)_{x\in[0,\infty[})$ of (E_x) is the growth rate of $x \mapsto \mathrm{rk}(\varphi_x)$.

A *morphism* of filtered directive systems from $(E_x)_{x\in[0,\infty[}$ to $(F_x)_{x\in[0,\infty[}$ consists of a positive number C and a family of linear maps $\Phi_x : E_x \longrightarrow F_{Cx}$ such that the following diagram commutes for all $0 \leq x_1 \leq x_2$:

$$\begin{tikzcd}
E_{x_1} \arrow[r, "\Phi_{x_1}"] \arrow[d] & F_{Cx_1} \arrow[d] \\
E_{x_2} \arrow[r, "\Phi_{x_2}"] & F_{Cx_2}
\end{tikzcd}$$

Two systems (E_x) and (F_x) are *isomorphic* if there exists a morphism (C, Φ) from (E_x) to (F_x) and a morphism (C', Ψ) from (F_x) to (E_x) such that for all $x \geq 0$

$$\Psi_{Cx} \circ \Phi_x = \varphi^E_{x,CC'x} \text{ and } \Phi_{C'x} \circ \Psi_x = \varphi^F_{x,CC'x}.$$

Lemma 4.3. *Two isomorphic filtered directive systems have the same growth rate.*

Proof. Consider two filtered directive systems (E_x) and (F_x). By definition, the following diagram

$$\begin{tikzcd}
E_{x_1} \arrow[r, "\varphi_{x_1}"'] \arrow[d] \arrow[dd, bend right=60, "\varphi_{x,CC'x}"'] & \lim E \arrow[d, "u"] \arrow[dd, bend right=90, looseness=2, "\mathrm{Id}"'] \\
F_{Cx_1} \arrow[r, "\psi_{Cx_1}"'] \arrow[d] & \lim F \arrow[d, "v"] \\
E_{CC'x_1} \arrow[r] & \lim E
\end{tikzcd}$$

commutes. Thus $\mathrm{rk}(\varphi_{x_1}) \leq \mathrm{rk}(\psi_{Cx_1})$. Similarly $\mathrm{rk}(\psi_{x_1}) \leq \mathrm{rk}(\varphi_{C'x_1})$. □

4.2. **Action filtration.** Let M be a compact manifold and α be a hypertight non-degenerate contact form on M. Fix $L > 0$ and let $C^{\mathrm{cyl}}_{\leq L}(M, \alpha)$ be the $\mathbb{Q}$-vector space generated by the good periodic Reeb orbits with period smaller than L. This is a finite dimensional vector space. Since the differential decreases the action, $(C^{\mathrm{cyl}}_{\leq L}(M, \alpha), \partial_{\leq L})_{L>0}$ is a chain complex. We denote $(HC^{\mathrm{cyl}}_{\leq L}(M, \alpha, J))_{L>0}$ the associated homology. The inclusion

$$i : C^{\mathrm{cyl}}_{\leq L}(M, \alpha) \longrightarrow C^{\mathrm{cyl}}_{\leq L'}(M, \alpha)$$

induces a linear map in homology for all $L' \geq L$. Similarly, given an set of free homotopy classes Λ and for all $L > 0$ we define a chain complex $(C^{\Lambda}_{\leq L}(M, \alpha), \partial_{\leq L})$ and a homology $HC^{\Lambda}_{\leq L}(M, \alpha, J)$.

Fact 4.4. *The families* $(HC^{cyl}_{\leq L}(M,\alpha,J))_{L>0}$ *and* $(HC^{\Lambda}_{\leq L}(M,\alpha,J))_{L>0}$ *are filtered directed systems whose morphisms are induced by inclusions. Besides*

$$\lim_{\to} HC^{cyl}_{\leq L}(M,\alpha,J) = HC^{cyl}_{*}(M,\alpha,J)$$
$$\lim_{\to} HC^{\Lambda}_{\leq L}(M,\alpha,J) = HC^{\Lambda}_{*}(M,\alpha,J).$$

Let M be a compact manifold and α be a non-degenerate contact form on M such that $(C_*(M,\alpha),\partial)$ admits an augmentation ε. Then ∂_1^ε decreases the action on $A^\varepsilon(V,\alpha)$ and we define a filtered directed systems.

Definition 4.5. The *growth rate* of contact homology is the growth rate of the associated filtered directed system.

Remark 4.6. As $\mathrm{rk}(\varphi_L) \leq \dim HC^{\mathrm{cyl}}_{\leq L}(M,\alpha,J) \leq \dim C^{\mathrm{cyl}}_{\leq L}(M,\alpha)$, if the growth rate of contact homology is exponential, the number of Reeb periodic orbits grows exponentially with the period.

4.3. **Invariance of the growth rate of contact homology.**

Fact 4.7. *The maps from Theorem 2.4 restrict to maps denoted* $\psi_{\leq L}$ *and* $\Psi_{\leq L}$ *in the filtered case. In addition* θ *and* Θ *restrict to maps*

$$\theta_{\leq L} : (A_{\leq L}(M,\alpha),\partial_J) \to (A_{\leq cL}(M,c\alpha),\partial_{J^c})$$
$$\Theta_{\leq L} : HC_{\leq L}(M,\alpha,J) \to HC_{\leq cL}(M,c\alpha,J^c).$$

Analogous restrictions exist in the cylindrical and linearized situations. The map $\varphi(\varepsilon_1,\varepsilon_2)$ *induces a map*

$$\Phi_{\leq L}(\varepsilon_1,\varepsilon_2) : HC^{\varepsilon_1}_{\leq L}(M,\alpha,J) \to HC^{\varepsilon_2}_{\leq L}(M,\alpha,J).$$

In addition to the properties from Theorem 2.4, these maps satisfies the following properties.

1. *For all* $0<c<1$,
$$\Theta_{\leq L}\left(c\alpha, J^c, \frac{1}{c}\right) \circ \Psi_{\leq L}((\alpha,J),(c\alpha,J^c))$$
is the map induced by the inclusion $HC_{\leq L}(\alpha,J) \to HC_{\leq \frac{L}{c}}(\alpha,J)$.
2. *If* $\varphi_1 = \psi((\alpha_1,J_1),(c\alpha,J^c)) \circ \psi((\alpha,J),(\alpha_1,J_1))$ *and* $\varphi_2 = \theta(\alpha,J,c)$ *then*
$$\Phi_{\leq \frac{L}{c}}(\varepsilon_2,\varepsilon_1) \circ \Theta^{\varepsilon_2}_{\leq L}\left(c\alpha, J^c, \frac{1}{c}\right) \circ \Psi^{\varepsilon_0}_{\leq L}((\alpha,J),(c\alpha,J^c))$$
is the morphism induced by the inclusion $HC^{\varepsilon_1}_{\leq L}(\alpha,J) \to HC^{\varepsilon_1}_{\leq \frac{L}{c}}(\alpha,J)$.

Proposition 4.8. *Let* α_0 *and* α_1 *be two homotopic hypertight contact forms on a compact manifold* M. *Under Hypothesis H, the two filtered directed systems* $HC^{cyl}_{\leq L}(M,\alpha_0)$ *and* $HC^{cyl}_{\leq L}(M,\alpha_1)$ *are isomorphic.*

Proof. The morphisms between $HC^{\mathrm{cyl}}_{\leq L}(M,\alpha_1)$ and $HC^{\mathrm{cyl}}_{\leq L}(M,\alpha_0)$ are

$$\varphi_L : HC^{\mathrm{cyl}}_{\leq L}(M,\alpha_1) \to HC^{\mathrm{cyl}}_{\leq \frac{L}{c}}(M,\alpha_0)$$
$$\varphi_L = \Theta^{\mathrm{cyl}}_{\leq L}\left(c\alpha_0, J_0^c, \frac{1}{c}\right) \circ \Psi^{\mathrm{cyl}}_{\leq L}\left((\alpha_1,J_1),(c\alpha_0,J_0^c)\right)$$

and

$$\varphi'_L : HC^{\mathrm{cyl}}_{\leq L}(M,\alpha_0) \to HC^{\mathrm{cyl}}_{\leq \frac{L}{c'}}(M,\alpha_1)$$
$$\varphi'_L = \Psi^{\mathrm{cyl}}_{\leq \frac{L}{c'}}\left(\left(\frac{\alpha_0}{c'}, J_0^{\frac{1}{c'}}\right),(\alpha_1,J_1)\right) \circ \Theta^{\mathrm{cyl}}_{\leq L}\left(\alpha_0, J_0, \frac{1}{c'}\right).$$

These morphisms give an isomorphism by Fact 4.7. $\square$

Corollary 4.9. *Let α_0 and α_1 be two homotopic hypertight contact forms on a compact manifold M. Under Hypothesis H, the associated cylindrical contact homologies have the same growth rate.*

Proposition 4.10. *Let α_0 and α_1 be two homotopic hypertight contact forms on a compact manifold M. Let Λ be a set of primitive free homotopy classes of M. Then the associated cylindrical partial contact homologies have the same growth rate.*

Proposition 4.11. *Let α_0 and α_1 be two isotopic contact forms, J_0 and J_1 be two adapted almost complex structures such that $(A_*(\alpha_0), \partial_{J_0})$ has an augmentation ε_0 and $\psi((\alpha_1, J_1), (\alpha_0, J_0))$ exists. Then, under Hypothesis H, the two filtered directed systems $HC^{\varepsilon_1}_{\leq L}(\alpha_1, J_1)$ and $HC^{\varepsilon_0}_{\leq L}(\alpha_0, J_0)$ are isomorphic. Thus, the growth rates of linearized contact homology are the same.*

Proof. Consider the morphisms

$$\varphi_L : HC^{\varepsilon_1}_{\leq L}(M, \alpha_1, J_1) \to HC^{\varepsilon_0}_{\leq L}(M, \alpha_0, J_0)$$
$$\varphi_L = \Psi^{\varepsilon_0}_{\leq L}\left((\alpha_1, J_1), (\alpha_0, J_0)\right)$$

and

$$\varphi'_L : HC^{\varepsilon_0}_{\leq L}(M, \alpha_0, J_0) \to HC^{\varepsilon_1}_{\leq \frac{L}{c}}(M, \alpha_1, J_1)$$
$$\varphi'_L = \Psi^{\varepsilon_1}_{\leq \frac{L}{c}}\left(\left(\frac{\alpha_0}{c}, J_0^{\frac{1}{c}}\right), (\alpha_1, J_1)\right) \circ \Phi_{\leq \frac{L}{c}}(\varepsilon_0^c, \varepsilon_0') \circ \Theta^{\varepsilon_0^c}_{\leq L}\left(\alpha_0, J_0, \frac{1}{c}\right)$$

where ε_0^c is the pull back augmentation of ε_0 by $\theta\left(\frac{1}{c}\alpha_0, J_0^{\frac{1}{c}}, c\right)$ and ε_0' is the pull back by $\psi((\alpha_1, J_1), (\alpha_0, J_0)) \circ \psi\left(\left(\frac{1}{c}\alpha_0, J_0^{\frac{1}{c}}, c\right), (\alpha_1, J_1)\right)$. These morphisms give an isomorphism by Fact 4.7. □

5. Positivity of intersection

Introduced by Gromov [28] and McDuff [40], positivity of intersection states that, in dimension 4, two distinct pseudo-holomorphic curves C and C' have a finite number of intersection points and that each of these points contributes positively to the algebraic intersection number $C \cdot C'$. In this text we will only consider the simplest form of positivity of intersection: let M be a 4-dimensional manifold, C and C' be two J-pseudo-holomorphic curves and $p \in M$ so that C and C' intersect transversely at p. Consider $v \in T_pC$ and $v' \in T_pC'$ two non-zero tangent vectors. Then (v, Jv, v', Jv') is a direct basis of T_pM (J orients T_pM). In contact world, positivity of intersection results in the following lemma.

Lemma 5.1. *Let (V, ξ) be a contact manifold, α be a contact form and J be an adapted almost complex structure. Consider U an open subset of $\mathbb{C}$, $u = (a, f) : U \to \mathbb{R} \times M$ a J-pseudo holomorphic curve and $p \in U$ such that $\mathrm{d}f_p$ is injective and transverse to $R(f(p))$.Then, $R(f(p))$ is positively transverse to $\mathrm{d}f_p$.*

Proof. Let $\gamma : [-\varepsilon, \varepsilon] \to V$ be a arc in a Reeb trajectory such that $\gamma(0) = f(p)$. Consider the holomorphic curve

$$\begin{array}{rcl} v : \mathbb{R} \times [-\varepsilon, \varepsilon] & \longrightarrow & \mathbb{R} \times V \\ (s, t) & \longmapsto & (s + f(p), \gamma(t)). \end{array}$$

The two holomorphic curves u and v intersect transversely at $u(p)$ and $\left(\frac{\partial}{\partial \tau}, R(f(p))\right)$ is a direct basis for the tangent plan to v at $u(p)$. The projection of u to M is smooth as $\mathrm{d}f_p$ is injective. The positivity of intersection gives the desired result. □

The hypothesis "$\mathrm{d}f_p$ injective and transverse to $R(f(p))$" is generic (see Theorem 2.1 and Proposition 2.2). We will use positivity of intersection in the following situation. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with a chart $J \times S^1 \times S^1$ and coordinates (x, y, z) such that $\alpha = f(x)\mathrm{d}\theta + g(x)\mathrm{d}z$. Assume that the tori $\{*\} \times S^1 \times S^1$ are incompressible[6] in M. Consider $u = (a, f) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ a pseudo-holomorphic cylinder with finite energy and asymptotic to γ_+ and γ_-. Assume that u intersects $\mathbb{R} \times \{x\} \times S^1 \times S^1 = \mathbb{R} \times T_r$ for all $x \in I$.

Lemma 5.2. *There exist a nonempty open interval $I \subset J$ such that for all $x_0 \in I$*

$$u^{-1}\left(u(\mathbb{R} \times S^1) \cap (\mathbb{R} \times T_{x_0})\right)$$

is a disjoint union of smooth circle homotopic to $\{*\} \times S^1$.

Let C be a circle given by Lemma 5.2, then C inherits the orientation of $\{*\} \times S^1$ and induces a homotopy class of T_{x_0}. Let p a vector tangent to T_{x_0} so that the associated line is in the homotopy class associated to C. If A is a collar neighborhood of C, denote $A_\pm$ the two connected components of $A \setminus C$ corresponding to the connected component of $\mathbb{R} \times S^1 \setminus C$ asymptotic to $\{\pm\infty\} \times S^1$.

Lemma 5.3. *If (p, R) is a direct basis of T_{x_0} then*

$$f(A_-) \subset]x, x_0[\times S^1 \times S^1 \text{ and } f(A_+) \subset]x_0, x[\times S^1 \times S^1.$$

Otherwise

$$f(A_-) \subset]x_0, x[\times S^1 \times S^1 \text{ and } f(A_+) \subset]x, x_0[\times S^1 \times S^1.$$

In other words, holomorphic cylinders cross a torus foliated by Reeb periodic orbits in just one direction.

Proof of Lemma 5.2. There exists an nonempty open interval $I' \subset I$ such that $I' \times S^1 \times S^1$ does not intersect γ_+ et γ_-. Thus $u^{-1}\left(u(\mathbb{R} \times S^1) \cap I \times S^1 \times S^1\right)$ is contained in a compact subset of $\mathbb{R} \times S^1$. As the points such that $\mathrm{d}u(s,t) = 0$ or $\frac{\partial}{\partial\tau} \in \mathrm{im}(\mathrm{d}u(s,t))$ are isolated in $\mathbb{R} \times S^1$, there exists an nonempty open interval $J \subset I'$ such that $J \times S^1 \times S^1$ does not contain images of points such that $\mathrm{d}u(s,t) = 0$ or $\frac{\partial}{\partial\tau} \in \mathrm{im}(\mathrm{d}u(s,t))$.

Consider $x_0 \in J$ and $(s,t) \in \mathbb{R} \times S^1$ such that $u(s,t) \in \mathbb{R} \times T_{x_0}$. As $\frac{\partial}{\partial\tau} \notin \mathrm{im}(\mathrm{d}u(s,t))$ and u is pseudo-holomorphic, $R(s,t) \notin \mathrm{im}(\mathrm{d}u(s,t))$ and

$$\mathrm{Vect}\left(\frac{\partial}{\partial\tau}, R_{u(s,t)}\right) \cap \mathrm{im}(\mathrm{d}u(s,t)) = \{0\}.$$

As $\mathrm{d}u(s,t) \neq 0$

$$\mathrm{Vect}\left(\frac{\partial}{\partial\tau}, R_{u(s,t)}\right) \oplus \mathrm{im}(\mathrm{d}_{(s,t)}u) = T_{u(s,t)}(\mathbb{R} \times M).$$

Thus $\mathrm{im}(\mathrm{d}u) + T(\mathbb{R} \times T_{x_0}) = T_u(\mathbb{R} \times M)$ and, by transversality,

$$u^{-1}\left(u(\mathbb{R} \times S^1) \cap (\mathbb{R} \times T_{x_0})\right)$$

is a 1-dimensional compact submanifold of $\mathbb{R} \times S^1$.

By contradiction, if $u^{-1}\left(u(\mathbb{R} \times S^1) \cap (\mathbb{R} \times T_{x_0})\right)$ has a contractible component C, then $u(C) = c$ is contractible in $\mathbb{R} \times V$. As $c \subset \mathbb{R} \times T_{x_0}$ and T_{x_0} is an incompressible torus, c is contractible in $\mathbb{R} \times T_{x_0}$. As $\mathrm{Vect}\left(\frac{\partial}{\partial\tau}, R_{u(s,t)}\right) \cap \mathrm{im}(\mathrm{d}u(s,t)) = \{0\}$, the projection of c to M is smooth and transverse to R. Yet the torus T_{x_0} is foliated by Reeb orbits. Thus $u^{-1}\left(u(\mathbb{R} \times S^1) \cap (\mathbb{R} \times T_{x_0})\right)$ has only non-contractible components and, as it is a smooth manifold, these components are homotopic to $\{*\} \times S^1$. ∎

[6] A torus T is *incompressible* in M if the map $\pi_1(T) \to \pi_1(M)$ is injective.

Proof of Lemma 5.3. Let $C(t)$ be a parametrization of C and c be the projection of $u(C)$ on T_{x_0}, c is a smooth curve transverse to R. If $\big(c'(t_0), R(c(t_0))\big)$ is a direct basis of T_{x_0} for some t_0, then $\big(c'(t), R(c(t))\big)$ is a direct basis for all t. Thus (p, R) is a direct basis T_{r_0} if and only if $\big(c'(t), R(c(t))\big)$ is a direct basis.

The sets $f(A_\pm)$ are connected and therefore contained in $]x_0, x[\times S^1 \times S^1$ or in $]x, x_0[\times S^1 \times S^1$. Let V be a normal vector to C at $C(t)$ so that $(V, C'(t))$ is a direct basis (V points toward A_+). Consider $v = \mathrm{d}f_{C(t)}V$, then $(v, c'(t), R)$ is a direct basis by positivity of intersection. If (p, R) is a direct basis then the x component of v is positive. Conversely if (R, p) is a direct basis, the x component of v is negative. $\square$

6. Contact homology of walled contact structures

In this section we prove Theorem 1.7. Walled contact structures are similar to contact structure $\ker(\sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z)$ on thickened tori near the wall and to S^1-invariant contact structures transverse to the fiber elsewhere. In the closed case, these situations can be studied with Morse-Bott theory (see Section 3). Theorem 1.7 states that cylindrical contact homology of a walled contact structure is the sum of cylindrical contact homologies of the components of this decomposition.

Let $\pi : M \to S$ be a circle bundle over a closed oriented surface and ξ be a contact structure on M walled by a curve Γ without contractible components. To prove Theorem 1.7 we first construct an "almost Morse-Bott" contact form such that

(1) in an union of thickened tori in a neighborhood of $\pi^{-1}(\Gamma)$, Reeb orbits foliate the tori;
(2) elsewhere, the Reeb vector field is tangent to the fiber.

This contact form is not of Morse-Bott type as some spaces N_T have a nonempty boundary. Then, in Section 6.2, we perturb the contact form from Section 6.1 as in the More-Bott case and control Reeb periodic orbits. We prove the quadratic growth rate of contact homology. In Section 6.3, we prove that there is no holomophic cylinders between two components of the decomposition using positivity of intersection and end the proof using Morse-Bott theory.

Remark 6.1. Contact structures $\ker(\sin(nx)\mathrm{d}y + \cos(nx)\mathrm{d}z)$ on $T^3 = T^2 \times S^1$ are walled by the curves $x = \frac{\pi}{2n} + \frac{k\pi}{n}$. Theorem 1.7 and Proposition 3.8 give the same contact homology.

6.1. "Almost Morse-Bott" contact form.

The following proposition results from Giroux's work [27].

Proposition 6.2. *Let ξ be a contact structure on a circle bundle M walled by a nonempty multi-curve Γ without contractible component. Then there exists a contact structure isotopic to ξ, a contact form α, and a neighborhood $U \simeq]-1, 1[\times \Gamma \times S^1$ of $\pi^{-1}(\Gamma) \simeq \{0\} \times \Gamma \times S^1$ with coordinates (x, y, z) such that:*

(1) $\ker(\alpha)$ *is walled by* Γ*;*
(2) $\alpha = \beta + \varepsilon \mathrm{d}z$ *on* $M \setminus U \simeq S_1 \times S^1$ *where* $\varepsilon = \pm 1$*;*
(3) $\alpha = f(x)\mathrm{d}y + g(x)\mathrm{d}z$ *on* U *where* f *is negative and strictly convex and* g *is increasing outside a neighborhood of* ± 1 *where* $g = \pm 1$ *and has an inflexion point at* 0*;*
(4) *the change of coordinates between* U *and a neighborhood of* $M \setminus U$ *is a linear map* $(x, y, z) \mapsto (x, y, z + ky)$.

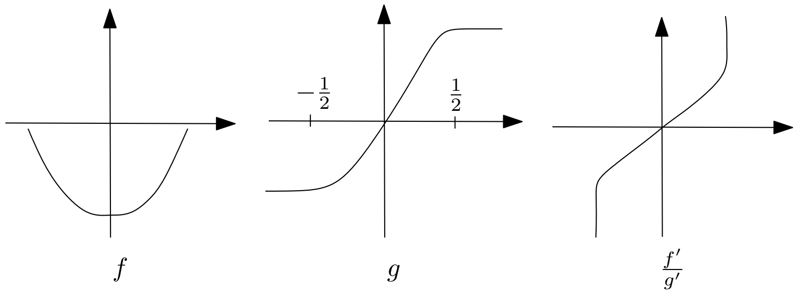


FIGURE 2. Maps f and g

Remark 6.3. On $M \setminus U$ the Reeb vector field is $\pm\frac{\partial}{\partial z}$. On U,

$$R_\alpha = \frac{1}{f'(x)g(x) - g'(x)f(x)} \begin{pmatrix} 0 \\ -g'(x) \\ f'(x) \end{pmatrix}.$$

The open set U is a union of thickened tori foliated by Reeb orbits.

Proof. Let $]-1,1[\, \times \Gamma \times S^1$ be a neighborhood of $\pi^{-1}(\Gamma) = \{0\} \times \Gamma \times S^1$ with coordinates (x, y, z) so that $\frac{\partial}{\partial x} \in \xi$ and S^1 is the fiber. In this chart, any contact form is written

$$\alpha = f(x, y, z)\mathrm{d}y + g(x, y, z)\mathrm{d}z$$

where $g(0, y, z) = 0$ and $g(x, y, z) \neq 0$ for all $x \neq 0$. Orient Γ so that Γ is negatively transverse to ξ. Without loss of generality, one can assume $f(0, y, z) = -1$.

Consider the path of contact forms

$$\alpha_s = \Big(sf(x, y, z) + (1-s)f(x, 0, 0)\Big)\mathrm{d}y + \Big(sg(x, y, z) + (1-s)g(x, 0, 0)\Big)\mathrm{d}z$$

in a small neighborhood of $\{0\} \times \Gamma \times S^1$. For all $s \in [0, 1]$, α_s is a contact form as $f(0, y, z) = f(0, 0, 0)$ and $g(0, y, z) = g(0, 0, 0)$. Moser's trick (see for instance [25, 2.2]) provides us with a vector field X_s near $\{0\} \times \Gamma \times S^1$ such that

(1) $X_s(0, y, z) = 0$;
(2) X_s is collinear to $\frac{\partial}{\partial x}$ (as $\xi_s \cap \xi = \mathbb{R}\frac{\partial}{\partial x}$);
(3) $\ker(\varphi_s^*\alpha) = \ker(\alpha_s)$ where φ_s is the flow of X_s.

Extend X_s to $V \times [0, 1]$ using a cut-off function. Then φ_s is well defined for all $s \in [0, 1]$. The contact structure associated to $\varphi_s^*\alpha$ is transverse to the fibers on $M \setminus \{0\} \times \Gamma \times S^1$ as $\varphi_s^*\alpha = f \circ \varphi_s \mathrm{d}y + g \circ \varphi_s \mathrm{d}z$, $(\varphi_s)_{|\{0\}\times\Gamma\times S^1} = \mathrm{Id}$ and $\alpha = \alpha_s$ on $M \setminus U$. Therefore ξ is isotopic to a contact structure such that there exist a contact form α and a chart $U' = \,]-1,1[\, \times \Gamma \times S^1$ near $\pi^{-1}(\Gamma)$ with

$$\alpha = f(x)\mathrm{d}y + g(x)\mathrm{d}z$$

where $g(0) = 0$ and $g(x) \neq 0$ for all $x \neq 0$. By the contact condition, $g'(0) > 0$ and one can assume that $g = -1$ on $\left]-1, \frac{1}{2}\right]$ and $g = 1$ on $\left[\frac{1}{2}, 1\right[$.

For each connected component of Γ, choose f_0 and g_0 so that

(1) f_0 is negative and strictly convex;
(2) g_0 is increasing outside a neighborhood of ± 1 where $g_0 = \pm 1$ and has an inflexion point at 0;
(3) $f_0 = f$ and $g_0 = g$ near $x = \pm 1$;
(4) $f_0(x)$ (resp. $g_0(x)$) is the same for all connected component and for all $x \in \left[-\frac{3}{4}, \frac{3}{4}\right]$.

Write $f(x)+ig(x) = \rho(x)\exp(i\theta(x))$ and $f_0(x)+ig_0(x) = \rho_0(x)\exp(i\theta_0(x))$. By the contact condition, θ and θ_0 are decreasing and have the same image as $g(x)=0$ (resp. $g_0(x)=0$) if and only if $x=0$. By use of Gray's theorem (see for instance [25, 2.2]) on the path

$$\Big((1-s)\rho(x)+s\rho_0(x)\Big)\exp\Big(i((1-s)\theta(x)+s\theta_0(x))\Big)$$

we obtain an isotopic contact form such that

$$\alpha = f_0(x)\mathrm{d}y + g_0(x)\mathrm{d}z$$

on U'. Let W be a neighborhood of a connected component of $M\setminus\Big(\,]-\frac{1}{2},\frac{1}{2}[\times\Gamma\times S^1\Big)$. As $\Gamma\neq\emptyset$ and S is connected, W is a manifold with boundary and the circle bundle is trivial. Let $S'\times S^1$ be a trivialization such that the change of coordinates between W and $]-1,1[\times\Gamma\times S^1$ is linear (i.e. $(x,y,z)\mapsto(x,y,z+ky)$) in polar coordinates near the boundary. Therefore $\alpha=\beta+\varepsilon\mathrm{d}z$ near ∂W. On W, $\alpha=\beta_z+h\mathrm{d}z$ and $h\neq 0$, so one can assume $h=\varepsilon$. By use of Gray's theorem on the path $\alpha_s = s\beta_z(x)+(1-s)\beta_0(x)+\varepsilon\mathrm{d}z$ we obtain the desired contact form. $\square$

6.2. **Morse-Bott perturbation.** Let α be a contact form as in Proposition 6.2 and write $U_i =]-\frac{1}{2},\frac{1}{2}[\times\Gamma_i\times S^1$ where $\Gamma=\cup\Gamma_i$. On U_i, $\alpha = f(x)\mathrm{d}y+g(x)\mathrm{d}z$ and the Reeb vector field is

$$R = \frac{1}{f'g-fg'}\begin{pmatrix} 0\\ -g'\\ f'\end{pmatrix}$$

If $f'(x)\neq 0$ and $\frac{-g'(x)}{f'(x)}=\frac{p}{q}$ with $p\wedge q=1$, the period of the Reeb periodic orbits in $T_x=\{x\}\times\Gamma_i\times S^1$ is

$$T = \left|\frac{(f'g-fg')q}{f'}\right| = |qg+fp|\,.$$

If $g'(x)\neq 0$ and $\frac{f'(x)}{-g'(x)}=\frac{q}{p}$ with $p\wedge q=1$, the period of the Reeb periodic orbits in T_x is

$$T = \left|\frac{(f'g-fg')p}{-g'}\right| = |qg+fp|\,.$$

In what follows we will assume $q\geq 0$. On $W=M\setminus\bigcup U_i$, all the fibers are periodic orbits of period 1.

As in the Morse-Bott case, let $\sigma(\alpha)$ denote set set of periods of Reeb periodic orbits, and write

$$N_L=\{p\in V,\varphi_L(p)=p\}\text{ and }S_L=N_L/S^1$$

for all $L\in\sigma(\alpha)$.

Lemma 6.4. *$\sigma(\alpha)$ is discrete and $\#(\sigma(\alpha)\cap[0,L])$ exhibits (exact) quadratic growth with L.*

Proof. There exist $A>0$ and intervals I_1 and I_2 such that

(1) $]-\frac{1}{2},\frac{1}{2}[=I_1\cup I_2$;
(2) $\left|\frac{1}{g'}\right|<A$ on I_1 and $\left|\frac{1}{f'}\right|<A$ on I_2;
(3) $\frac{1}{A}<f'g-fg'<A$, $|f'|<A$ and $|g'|<A$.

Consider $L>0$ and $x\in I_1$ such that $\frac{f'(x)}{-g'(x)}=\frac{q}{p}$ and $\left|\frac{(f'g-fg')p}{-g'(x)}\right|<L$. There is at most $3A^6L^2$ rational numbers $\frac{q}{p}$ such that $\left|\frac{(f'g-fg')p}{-g'(x)}\right|<L$ for some $x\in I_1$.

As $x \mapsto \frac{f'(x)}{-g'(x)}$ is increasing, for all rational number $\frac{q}{p}$ there is one x such that $\frac{f'(x)}{-g'(x)} = \frac{q}{p}$. Therefore the growth rate is at most quadratic.

The growth rate is also at least quadratic: consider $B > 0$ and p, q such that $p^2 + q^2 \leq B$. Then, there exists x such that $\frac{f'(x)}{-g'(x)} = \frac{q}{p}$. The associated torus is foliated by Reeb periodic orbits of period smaller than $A^2 B$. $\square$

Fact 6.5. *Let a be a loop such that $[a] \neq [\text{fiber}]^k$ and $[a] \neq [\text{fiber}]^k[\Gamma_j]^{k'}$, then there exists a contact form α such that $HC_*^{[a]}(V, \alpha) = 0$.*

As in the Morse-Bott case, we perturb the degenerate contact form with Morse functions on space S_L. Fix $L > 0$ and for all $L' \leq L$ consider a function $f_{L'} : N_{L'} \to \mathbb{R}$ such that

(1) if $T_x \subset N_{L'}$, then $f_{L'}(x, y, z) = h(qy - pz)$ where $h : S^1 \to \mathbb{R}$ is a Morse function;
(2) f_1 is z-invariant on W, f_1 does not depend on y and $\varepsilon \frac{\partial f_1}{\partial x} > 0$ in cylindrical coordinates (x, y, z) near ∂W,.

Extend $f_{L'}$ to M by use of a cut-off function. Let $\overline{f}_L$ denote the sum of $f_{L'}$ for all $L' \leq L$. Perturb the contact equation in

$$\alpha_{\lambda,L} = (1 + \lambda \overline{f}_L)\alpha.$$

Note that the length in the x-coordinate of connected components of $\operatorname{dom}(\overline{f}_L)$ tends to 0 as $N \to \infty$ and that the flow of $R_{\alpha_{L,\lambda}}$ preserves $\operatorname{dom}(\overline{f}_L)$ and $V \setminus \operatorname{dom}(\overline{f}_L)$.

Lemma 6.6. *For all $L > 0$, there exists $\Lambda > 0$ such that for all $0 < \lambda \leq \Lambda$, the periodic orbits of $\alpha_{\lambda,L}$ with period smaller than L correspond to critical points of $f_{L'}$ on $S_{L'}$, $L' \in \sigma(\alpha) \cap [0, L]$. These periodic orbits are non-degenerate.*

Proof. Outside a neighborhood of ∂W, Morse-Bott theory applies directly (see [3, Lemma 2.3]). In a neighborhood of ∂W, in the trivializing chart of W with coordinates (x, y, z), contact equation is written

$$\alpha = (f(x) + kg(x))\mathrm{d}y + g(x)\mathrm{d}z = f_W(x)\mathrm{d}y + g(x)\mathrm{d}z.$$

as the change of coordinates is linear (Proposition 6.2). As $\overline{f}_L$ only depends on x,

$$R_{\alpha_{L,\lambda}} = \frac{1}{(f'_W g - f_W g')(1 + \lambda \overline{f}_L)^2} \begin{pmatrix} 0 \\ -g'(1 + \lambda \overline{f}_L) - \lambda g \overline{f}'_L \\ f'_W(1 + \lambda \overline{f}_L) + \lambda f_W \overline{f}'_L \end{pmatrix}.$$

In a small neighborhood of ∂W and for λ small enough, the y coordinate of the Reeb vector field is as small as desired and do not vanish. Therefore there is no Reeb periodic orbit with period smaller that L. $\square$

Lemma 6.7. *Let a be a loop such that $[a] = [\text{fiber}]^k$ with $k \neq 0$ or $[a] = [\text{fiber}]^k[\Gamma_{j_0}]^{k'}$ with $k' \neq 0$. Then, there exist L_0 and $L \mapsto \lambda(L) > 0$ decreasing such that for all $L \geq L_0$ and $\lambda \leq \lambda(L)$*

(1) Reeb periodic orbits of $\alpha_{L,\lambda}$ homotopic to a have a period smaller than L;
(2) $\alpha_{L,\lambda}$ is hypertight

In addition, there exist arbitrarily small non-degenerate and hypertight perturbations of $\alpha_{L,\lambda}$.

Proof. Let $W_1, \dots, W_m$ be the connected components of W. Consider $\bigcup_{i=1}^m W'_i \cup \bigcup_{j=1}^n U'_i$ an open covering of M such that $W'_i \cap \pi^{-1}(\Gamma) = \emptyset$ for all $i = 1 \dots m$ and $U'_j \cap W = \emptyset$ for all $j = 1 \dots n$. There exists $\varepsilon > 0$ such that, in the trivialization of W'_i induced by Proposition 6.2, $|R_z| > \varepsilon$ where R_z is the z-coordinate of the Reeb vector field and, in the trivialization of U'_j induced by Proposition 6.2, $|R_y| > \varepsilon$

where R_y is the y-coordinate of the Reeb vector field. If there exists a loop b in W_i (resp. U_j) such that $[b] = [a]$, let k_i (resp. k'_j) denote the multiplicity of the fiber (resp. Γ_j) in the decomposition of b in the associated trivialization. Consider $L_0 > 0$ such that

(1) $L_0 > \dfrac{\max(\{|k_i|, |k'_j|\})}{\varepsilon}$;
(2) periodic orbits of R_α homotopic to a have a period smaller than L_0;
(3) for all $L'' \geq L_0$, the connected components of $\mathrm{dom}(f_{L''})$ are contained in an open component of the covering.

By Lemma 6.6, there exists $\lambda(L)$ such that for all $\lambda \leq \lambda(L)$, all Reeb periodic orbits of $\alpha_{L,\lambda}$ with period smaller that L are non-degenerate and such that ε is a lower bound for the z-component of $R_{\alpha_{L,\lambda}}$ in W'_i and for the y-component in U'_j.

Let γ be a $\alpha_{L,\lambda}$ Reeb periodic orbit with period greater than L. Then, either $\gamma \subset \mathrm{dom}(f_L)$ or $\gamma \subset M \setminus \mathrm{dom}(f_L)$. If $\gamma \subset \big(M \setminus \mathrm{dom}(f_L)\big)$ then γ is not homotopic to a by condition 2. If $\gamma \subset \mathrm{dom}(f_L)$, by condition 3, either $\gamma \subset \big(\mathrm{dom}(f_L) \cap W'_i\big)$ or $\gamma \subset (\mathrm{dom}(f_L) \cap U'_j)$. If $\gamma \subset \big(\mathrm{dom}(f_L) \cap W'_i\big)$, then γ covers the fiber at least $\pm\varepsilon L$ times and hence γ covers the fiber at least $|k_i|+1$ or $-|k_i|-1$ times by condition 1. If $\gamma \subset (\mathrm{dom}(f_L)\cap U'_j)$ then covers Γ_j at least $|k'_j|+1$ or $-|k'_j|-1$ times. Consequently, γ is not homotopic to a and non-contractible as Γ_j is not contractible and the fiber is not a torsion element. By Lemma 6.6, $\alpha_{L,\lambda}$ is hypertight.

Assume that the boundaries of U'_j and W'_i are tori $x = \mathrm{cst}$ with dense Reeb orbits. To get a non-degenerate hypertight perturbation of $\alpha_{L,\lambda}$, we choose a small non-degenerate perturbation that preserves the boundaries of U'_j and W'_i. □

Lemma 6.8. *Under Hypothesis H, the growth rate of contact homology is (at most) quadratic.*

Proof. Let α' be a non-degenerate and hypertight contact form (given for instance by Lemma 6.7). Let α_{L_i,λ_i} be a sequence of contact forms with $L_i \to \infty$ such that $L_i \notin \sigma(\alpha)$ and $\lambda_i \leq \lambda_1(L_i)$ for all $i \in \mathbb{N}^*$. Perturb α_{L_i,λ_i} to obtain a non-degenerate hypertight form α'_{L_i,λ_i} (Lemma 6.7). For λ_i small enough and for small perturbations, the Reeb periodic orbits of α'_{L_i,λ_i} with period smaller than L_i are in bijection with the Reeb periodic orbits of α_{L_i,λ_i} with period smaller than L_i and the difference between their period and the period of the associated R_α periodic orbits is bounded by $\frac{1}{2}$. Thus, there exists $C > 0$ such that for all $i \in \mathbb{N}^*$ and for all $L \leq L_i$

$$\#C^{\mathrm{cyl}}_{\leq L}(V, \alpha'_{L_i,\lambda_i}) \leq C\#\left(\sigma(\alpha) \cap [0, L+1]\right).$$

In addition, there exists $D > 0$ such that

$$\frac{1}{D} < \sup\left\{f_{L_i,\lambda_i}(p), \frac{1}{f_{L_i,\lambda_i}(p)}\right\} < D$$

where $\alpha' = f_{L_i,\lambda_i}\alpha'_{L_i,\lambda_i}$.

By invariance of cylindrical contact homology (Corollary 4.9) and by [17, 10], there exists $C(D)$ such that, for all $L > 0$ and for all i, $\mathrm{rk}(\psi_L) \leq \mathrm{rk}(\psi^i_{C(D)L})$ where $\psi^i_L : HC^{\mathrm{cyl}}_{\leq L}(M, \alpha'_{L_i,\lambda_i}) \to HC^{\mathrm{cyl}}(M, \alpha'_{L_i,\lambda_i})$ and $\psi_L : HC^{\mathrm{cyl}}_{\leq L}(M, \alpha') \to HC^{\mathrm{cyl}}(M, \alpha')$ are the maps defining the direct limit. Hence,

$$\mathrm{rk}(\psi_L) \leq C\#\left(\sigma(\alpha) \cap [0, C(D)L + 1]\right)$$

and $\mathrm{rk}(\psi_L)$ exhibits a quadratic growth. □

6.3. **Holomorphic cylinders and Morse-Bott theory.** Let $[a]$ be a free homotopy class such that $[a] = [\text{fiber}]^k$ or $[a] = [\text{fiber}]^k[\Gamma_j]^{k'}$. By Lemma 6.7 there exists $L > 0$ and $\lambda > 0$ such that all the $R_{\alpha_{L,\lambda}}$-periodic orbits homotopic to a are non-degenerate, associated a critical point of $f_{L'}, L' \leq L$ and have a period smaller than L. Consider $W'_j = \bigcup_{k \in K_j} U_k$ where $K_j = \{k, U_k \text{ is adjacent to } W_j\}$. Then W'_j is a trivial circle bundle. Extend the trivialization from Proposition 6.2 in $W'_j \simeq S'_j \times S^1$. In these coordinates, $\alpha = (f(x) + mg(x))\,\mathrm{d}y + g(x)\mathrm{d}z$ and the Reeb vector field is positively collinear to

$$R_\alpha = \begin{pmatrix} 0 \\ -g' \\ f' + mg' \end{pmatrix}.$$

Note that the y-coordinate is negative in $W'_j \setminus W_j$.

Lemma 6.9. *Let $u = (a_u, f_u) : \mathbb{R}\times S^1 \to \mathbb{R}\times M$ be a holomophic cylinder negatively asymptotic to $\gamma \in W_j$. Then $f_u(\mathbb{R} \times S^1) \subset W_j$.*

Proof. We prove the lemma by contradiction. Assume $f_u(\mathbb{R} \times S^1) \cap U_l \neq \emptyset$ for $l \in K_j$, then there exists an open interval I such that, in $I \times S^1 \times S^1 \subset U_l$,

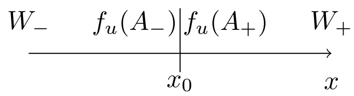
(1) $\alpha = f_1(x)\mathrm{d}\theta + g_1(x)\mathrm{d}z$ in the trivialization of W'_j;
(2) $f_u(\mathbb{R} \times S^1) \cap \{x\} \times S^1 \times S^1 \neq \emptyset$ for all $x \in I$.

By Lemma 5.2, for all $x_0 \in I$,

$$u^{-1}\left(u(\mathbb{R} \times S^1) \cap \mathbb{R} \times T_{x_0}\right)$$

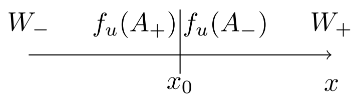
is a finite union of smooth circles homotopic $\{*\} \times S^1$. For all $l \in K_j$, choose $x_0 \in I$ and cut $\mathbb{R} \times S^1$ along the associated circles. Choose the connected component asymptotic to $-\infty \times S^1$. Let C denote the oriented boundary of this component and choose a collar neighborhood $A = \overline{A_+} \cup \overline{A_-}$ of C as in Lemma 5.3: $A_\pm$ are open annuli in the connected component of $\mathbb{R} \times S^1 \setminus C$ asymptotic to $\pm\infty \times S^1$. Let W_+ (resp W_-) denote the union of the connected components of W such that the Reeb vector field is positively (resp negatively) tangent to the fiber.

If $\gamma \subset W_+$, the line in T_{x_0} tangent to $p = (0, 1)$ is in the homotopy class of γ. Hence (p, R) is a direct basis. By Lemma 5.3, $f_u(A_-) \subset]x, x_0[\times S^1 \times S^1$.



Yet $f_u(A_-) \subset]x_0, x[\times S^1 \times S^1$ as u is negatively asymptotic to γ. This leads to a contradiction.

If $\gamma \subset W_-$, the line tangent to $p = (0, -1)$ in T_{x_0} is in the homotopy class of γ and (p, R) is not a direct basis. Thus $f_u(A_-) \subset]x_0, x[\times S^1 \times S^1$.



This leads to a contradiction as u is negatively asymptotic to γ. □

Lemma 6.10. *Let $u = (a_u, f_u) : \mathbb{R}\times S^1 \to \mathbb{R}\times M$ a holomophic cylinder negatively asymptotic to $\gamma \in U_j$. Then $f_u(\mathbb{R} \times S^1) \subset U_j$ and $f_u(\mathbb{R} \times S^1) \subset \mathrm{dom}(\overline{f_T})$*

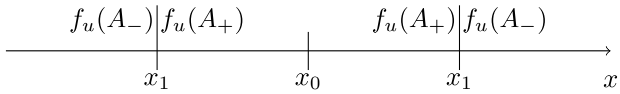
Proof. Consider x_0 such that $\gamma \in T_{x_0}$ in the trivialization $\left]-\frac{1}{2}, \frac{1}{2}\right[\times S^1 \times S^1$ of U_j. We prove the lemma by contradiction. Thus there exists an open interval

$I \subset \left]-\frac{1}{2}, \frac{1}{2}\right[$ such that $\alpha = f(x)\mathrm{d}y + g(x)\mathrm{d}z$ and $f(\mathbb{R}\times S^1) \cap \{x\} \times S^1 \times S^1 \neq \emptyset$ for all $x \in I$. By Lemma 5.2, for all $x_1 \in I$,

$$u^{-1}\left(u(\mathbb{R}\times S^1) \cap \mathbb{R} \times T_{x_1}\right)$$

is a finite union of smooth circles homotopic to $\{*\} \times S^1$. Cut $\mathbb{R}\times S^1$ along these circles and denote C the oriented boundary of the component asymptotic to $-\infty \times S^1$. Let p be such that the line tangent to p in T_{x_0} is homotopic to γ and A be a collar neighborhood $A = \overline{A_+} \cup \overline{A_-}$ of C as in Lemma 5.3: $A_\pm$ are open annuli in the connected component of $\mathbb{R}\times S^1 \setminus C$ asymptotic to $\pm\infty \times S^1$.

If $x_1 > x_0$ then (p, R) is not a direct basis (f' is increasing) and $f_u(A_-) \subset]x_1, x[\times S^1 \times S^1$ by Lemma 5.3. If $x_1 < x_0$ then (p, R) is a direct basis and $f_u(A_-) \subset]x, x_1[\times S^1 \times S^1$.



This leads to a contradiction as u is negatively asymptotic to γ. $\square$

Lemma 6.11. *For all $j = 1 \dots m$, there exists a contact closed manifold without boundary $(\tilde{W}_j, \tilde{\alpha})$ extending (W_j, α) such that $\tilde{\alpha}$ is of Morse-Bott type. For all $i = 1 \dots n$, there exists a contact closed manifold without boundary $(\tilde{U}_i, \tilde{\alpha})$ extending (U_i, α) such that $\tilde{\alpha}$ is of Morse-Bott type.*

Proof. In the trivialization $W_j \simeq S_j \times S^1$, the contact form is $\alpha = \beta + \varepsilon\mathrm{d}z$ and, near ∂W_j, there exists coordinates $(x, y, z) \in [0,1] \times S^1 \times S^1$ such that $\{1\} \times S^1 \times S^1 \subset \partial W_j$ and $\alpha = f(x)\mathrm{d}y + \varepsilon\mathrm{d}z$.

Let S' be an oriented compact surface such that $\partial S'$ and ∂S_j have the same number of connected components. Choose a pairing between these components and glue a neighborhood of each component of ∂W_j to a neighborhood of the associated component of $\partial S' \times S^1$ with the diffeomorphism $\varphi : (x, y, z) \mapsto (x, y, z + ey)$ where $e \in \mathbb{Z}$. Let $\tilde{W}_j$ denote the resulting manifold. Near $\partial S' \times S^1$,

$$\varphi^*\alpha = \big(f(x) + e\varepsilon\big)\mathrm{d}y + \varepsilon\mathrm{d}z = \tilde{\beta}_e + \varepsilon\mathrm{d}z.$$

For each component, choose e so that $\varepsilon\tilde{\beta}_e$ is positive on $\partial S'$. There exist a 1-form β' on S' such that $\varepsilon\mathrm{d}\beta' > 0$ and $\tilde{\beta}_e = \beta'$ near the boundary. The contact form $\beta' + \varepsilon\mathrm{d}z$ extends $\varphi^*\alpha$ and the induced form $\tilde{\alpha}$ on $\tilde{W}_j$ is of Morse-Bott type.

On $U_j = A \times S^1$, the contact form is written $\alpha = f(x)\mathrm{d}y + g(x)\mathrm{d}z$. Extend f and g to maps $\tilde{f}$ and $\tilde{g}$ on S^1 so that $\tilde{\alpha} = \tilde{f}(x)\mathrm{d}y + \tilde{g}(x)\mathrm{d}z$ is a contact form on T^3. The form $\tilde{\alpha}$ is of Morse-Bott type. $\square$

Proof of Theorem 1.7. It remains to compute contact homology when $[a] = [\text{fiber}]^k$ or $[a] = [\text{fiber}]^k[\Gamma_j]^{k'}$. By Lemma 6.7, there exist $L > 0$ and $\lambda > 0$ such that all the $R_{\alpha_{L,\lambda}}$-periodic orbtis homotopic to a have a period smaller than L and are non-degenerate and associated to a critical point of $f_{L'}$, $L' \leq L$.

Extend $\overline{f}_L$ to the contact manifolds $\tilde{W}_j$ and $\tilde{U}_i$ (Lemma 6.11) to get a Morse-Bott perturbation. Let (λ_n) be a decreasing sequence such that $\lambda_n \in]0, \lambda]$ and $\lim_{n\to\infty} \lambda_n = 0$. Choose almost complex structures J_{λ_n} adapted to $(V, \alpha_{L,\lambda_n})$ and $\tilde{J}_{\lambda_n}$ adapted to the union of $(\tilde{W}_j, \tilde{\alpha}_{L,\lambda_n})$ for $j = 1 \dots m$ and $(\tilde{U}_i, \tilde{\alpha}_{L,\lambda_n})$ for $i = 1 \dots i$ so that

(1) $J_{\lambda_n} = \tilde{J}_{\lambda_n}$ on $\mathbb{R} \times W_j$ and $\mathbb{R} \times U_i$;
(2) J_{λ_n} and $\tilde{J}_{\lambda_n}$ are S^1-invariant on $\mathbb{R} \times N_{L'}$ for all $L' \leq L$;
(3) $(f_{L'}, g_{L'})$ (resp $(\tilde{f}_{L'}, \tilde{g}_{L'})$) is Morse-Smale on $S_{L'}$ where $g_{L'}$ (resp $\tilde{g}_{L'}$) is the metric induced by J_{λ_n} (resp $\tilde{J}_{\lambda_n}$) for all $L' \leq L$.

By Theorems 3.4 and 3.5, for all $j = 1 \dots m$ (resp $i = 1 \dots n$) and for n big enough, $\tilde{J}_{\lambda_n}$-holomorphic cylinders asymptotic Reeb periodic orbits in W_j (resp. U_i) are contained in W_j (resp. U_i) as gradient lines between two points in S_j are contained in S_j.

By Lemmas 6.9 and 6.10, for all $j = 1 \dots m$ (resp $i = 1 \dots n$) and for n big enough, J_{λ_n}-holomorphic cylinders asymptotic Reeb periodic orbits in W_j (resp. U_i) are contained in W_j (resp. U_i). Therefore the differential of contact homology is well defined and identifies with the differential in the Morse-Bott case and thus with the differential in Morse homology. Hence

(1) if $[a] = [\text{fiber}]^k$ with $\pm k > 0$,
$$HC_*^{[a]}(M, \alpha, \mathbb{Q}) = \bigoplus_{W_j \subset W_\pm} H_*^{\mathcal{M}}(W_j, (f_1, g_1), \mathbb{Q})$$

(2) si $[a] = [\text{fiber}]^k [\Gamma_j]^{k'}$ with $k' \neq 0$,
$$HC_*^{[a]}(V, \alpha, \mathbb{Q}) = \bigoplus_{[\Gamma_i] = [\Gamma_j]} H_*^{\mathcal{M}}(S^1, (f_L, g_L), \mathbb{Q})$$

where $H_*^{\mathcal{M}}(X, (f, g), \mathbb{Q})$ is the Morse homology associated to the function f and the metric g (we do not consider the graduation in the identifications). □

Latschev and Wendl [38] used similar methods to study algebraic torsion of contact homology.

7. Hyperbolicity and exponential growth rate

In this section we prove Theorem 1.2. This results hinges on the exponential growth of contact homology for a specific family of contact structures (Theorem 1.3). The invariance of contact homology leads to the exponential growth of $N_L(\alpha)$ for all contact forms for non-degenerate contact forms. For a general non-degenerate contact form, the proof depends on Hypothesis H. Yet, as we restrict contact homology to primitive homotopy classes, the proof of invariance of cylindrical contact homology may be easier than in the general case.

Let M be a 3-manifold which can be cut along a nonempty family of incompressible tori $T_1, \dots T_N$ into irreducible manifolds including a hyperbolic component that fibers on the circle. We construct contact forms on each irreducible components and add torsion near the irreducible tori $T_k, k = 1 \dots N$ (Section 7.2). We compute the growth rate of contact homology by controlling the holomorphic cylinders that intersect the tori $T_k, k = 1 \dots N$ (Section 7.3). The study of periodic orbits and contact homology in the hyperbolic component hinges on properties of periodic points of pseudo-Anosov automorphisms recalled in Section 7.1.

7.1. **Periodic points of pseudo-Anosov automorphisms.**

Let S be a compact orientable surface. An automorphism $\psi : S \to S$ is said to be *pseudo-Anosov* if there exists two measured foliations $(\mathscr{F}^s, \mu^s)$ and $(\mathscr{F}^u, \mu^u)$ such that $\psi(\mathscr{F}^u, \mu^u) = (\mathscr{F}^u, \lambda^{-1}\mu^u)$ and $\psi(\mathscr{F}^s, \mu^s) = (\mathscr{F}^s, \lambda\mu^s)$ for a positive real number λ. One can refer to [11] for the study of pseudo-Anosov on surfaces without boundary and [22] for a complete presentation. In this section, we assume that S is compact and $\partial S = \emptyset$.

Theorem 7.1. *The number of simple k-periodic points of a pseudo-Anosov automorphism on S exhibits an exponential growth with k.*

This theorem follows from the construction of a Markov partition on S (see [17, 11]). Nielsen classes are used to transfer properties of periodic points of a pseudo-Anosov to properties of periodic points of homotopic diffeomorphisms.

Definition 7.2. Let $h : S \to S$ be an automorphism. Two fixed points x and y are *in the same Nielsen class* if there exists $\delta : [0,1] \to S$ continuous such that $\delta(0) = x$, $\delta(1) = y$ and $h(\delta)$ is homotopic to δ.

Let $h_t : S \to S, t \in [0,1]$ be a homotopy of automorphism of S. A fixed points x_0 of h_0 and a fixed points x_1 of h_1 are *in the same Nielsen class* if there exists $\delta : [0,1] \to S$ continuous such that $\delta(0) = x$, $\delta(1) = y$ and $t \mapsto h_t(\delta(t))$ is homotopic to δ.

See [23] for more information on Nielsen classes. These definitions extend naturally to periodic points. Two periodic points are in the same Nielsen class of a diffeomorphism h if and only if the induced periodic orbits of the vertical vector field in the suspension of S by h are homotopic.

Theorem 7.3. *All the periodic points of a pseudo-Anosov automorphism on S are in different Nielsen classes.*

A periodic point x of $h : S \to S$ is *non-degenerate* is 1 is not an eigenvalue of $\mathrm{d}h^k(x)$. For a non-degenerate periodic point, let $\varepsilon_{h^k}(x)$ denote the sign of $\det(\mathrm{d}h^k(x) - \mathrm{Id})$. If all the periodic points in a Nielsen class n are non-degenerate, consider

$$\Lambda_{h^k}(n) = \sum_{x \in n} \varepsilon_{h^k}(x).$$

Theorem 7.4. *Let $h_0 : S \to S$ and $h_1 : S \to S$ be two homotopic automorphisms, x_0 be a periodic point of h_0 and x_1 be a periodic point of h_1 in the same Nielsen class. If the Nielsen classes n_0 of x_0 (for h_0) and n_1 of x_1 (for h_1) contain only non-degenerate points then $\Lambda_{h_0^k}(n_0) = \Lambda_{h_1^k}(n_1)$.*

Theorem 7.5. *Let $S_1 = S \setminus \cup_{i=1}^{I} D_i$ where for all $i = 1 \dots I$, D_i are disjoint open disks in S. Let $h : S_1 \to S_1$ be an automorphism such that $h = \mathrm{Id}$ in a neighborhood of ∂S_1 and $\psi : S_1 \to S_1$ be a pseudo-Anosov automorphism homotopic to h. Extend h to S by the identity and let $\hat{h}$ denote the resulting automorphism. Then, there exists a branched cover $\hat{S}$ of S and a pseudo-Anosov $\hat{\psi}$ homotopic to $\hat{h}$ such that the projection of $\hat{\psi}$ is ψ.*

7.2. **Contact forms on M.**

7.2.1. *In the hyperbolic component.* Let M_0 be the hyperbolic component, then M_0 is written $(S \times S^1)/h$ where

(1) S is a compact oriented surface with boundary;
(2) $h : S \to S$ is a diffeomorphism homotopic to a pseudo-Anosov;
(3) $h = \mathrm{Id}$ in a neighborhood of ∂S.

We use the usual construction on a contact structure on a suspension. Choose cylindrical coordinates (r, θ) in a neighborhood of ∂S so that $\frac{\partial}{\partial \theta}$ is positively tangent to ∂S. Let β be a 1-form on S such that $\mathrm{d}\beta > 0$ and $\beta = b(r)\mathrm{d}\theta$ with $b > 0$ and $b' > 0$ near ∂S. Let $F : [0,1] \to [0,1]$ be a smooth non-decreasing function such that $F = 0$ near 0 and $F = 1$ near 1. On $S \times [0,1]$ consider the contact form

$$\alpha = (1 - F(t))\beta + F(t)h^*\beta + \mathrm{d}t$$

where t is the coordinate on $[0,1]$. This contact form induces a contact form on M_0. The associated contact structure is universally tight.

Lemma 7.6. *The Reeb vector field is positively transverse to $S \times \{*\}$ and the first return map on $S \times \{0\}$ is homotopic to h.*

Proof. If the Reeb vector field is tangent to $S \times \{t\}$ in (p,t) then

$$\iota_R((1-F(t))\mathrm{d}\beta + F(t)h^*\mathrm{d}\beta)(p,t) = 0$$

as

$$\mathrm{d}\alpha = (1-F(t))\mathrm{d}\beta + F(t)h^*\mathrm{d}\beta + F'(t)\mathrm{d}t \wedge (h^*\beta - \beta).$$

Yet $\mathrm{d}\beta$ and $h^*\mathrm{d}\beta$ are two positive volume forms. Hence R is transverse to $S \times \{t\}$. It is positively transverse by the boundary condition. The first return map is well defined and homotopic to h as h is the first return map of $\frac{\partial}{\partial t}$ on $S \times \{0\}$ and R and $\frac{\partial}{\partial t}$ are homotopic in the space of vector fields transverse to $S \times \{*\}$. □

In M_0, Reeb periodic orbits correspond to periodic points of the first return map on $S \times \{0\}$. Without loss of generality, we may assume that all the periodic points of the first return map in the interior of S are non-degenerate.

7.2.2. *In non-hyperbolic components.* We use the following theorem of Colin and Honda.

Theorem 7.7 (Colin-Honda, [16])**.** *Let M be a compact, oriented, irreducible 3-manifold with boundary such that ∂M is a union of tori. Then there exists an hypertight contact form α on M such that, in a neighborhood $T \times I$ with coordinates (x,y,z) of each boundary components, $\alpha = \cos(z)\mathrm{d}x - \sin(z)\mathrm{d}y$. In addition there exists arbitrarily small non-degenerate hypertight perturbations of α.*

The construction in [16] gives the same contact structures as [34] and [15]. Without loss of generality, all the periodic orbits whose free homotopy class does not correspond to a class in the boundary are non-degenerate.

7.2.3. *Interpolation and torsion.* In the previous sections we constructed an hypertight contact form α on $M \backslash \bigcup_{k=1}^{N} \nu(T_k)$ where $\nu(T_k)$ is a neighborhood of T_k. Choose $k \in 1 \dots N$. There exists coordinates (x,y,z) in a neighborhood $T_k \times [a,b]$ of T_k such that in a neighborhood of $T \times \{a\}$ the contact form is written $f_a(x)\mathrm{d}y + g_a(x)\mathrm{d}z$ and in a neighborhood of $T \times \{b\}$ the contact form is written $f_b(x)\mathrm{d}y + g_b(x)\mathrm{d}z$.

Lemma 7.8. *For all $n \in \mathbb{N}^*$, there exists $f_n : [a,b] \longrightarrow \mathbb{R}$ and $g_n : [a,b] \longrightarrow \mathbb{R}$ two smooth functions such that*

(1) *f_n extends f_a and f_b;*
(2) *g_n extends g_a and g_b;*
(3) *$\alpha = f_n(x)\mathrm{d}y + g_n(x)\mathrm{d}z$ is a contact form;*
(4) *in coordinates (θ, z), the Reeb vector field R_α sweeps out an angle in*

$$\left]2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right].$$

Proof. The contact condition is $f_n' g_n - f_n g_n' > 0$ and the Reeb vector field is

$$R_\alpha = \frac{1}{f_n' g_n - f_n g_n'} \begin{pmatrix} 0 \\ -g_n' \\ f_n' \end{pmatrix}.$$

The conditions 3 and 4 are equivalent to "the parametrized curve (f_n, g_n) in $\mathbb{R}^2$ turns clockwise and its normal vector sweeps out an angle in $\left]2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$". We choose a parametric curve in $\mathbb{R}^2$ extending (f_a, g_a) and (f_b, g_b) with these properties. □

For all $n \in \mathbb{N}^*$, construct a contact form α_n on M by extending α by $\alpha_n = f_n(x)\mathrm{d}y + g_n(x)\mathrm{d}z$ in a neighborhood of T_1 and by $\alpha_n = f_1(x)\mathrm{d}y + g_1(x)\mathrm{d}z$ in a neighborhood of $T_2, \dots T_N$.

Remark 7.9. If $\{b\} \times T$ is in ∂M_0, then $f_b < 0$, $f_b' > 0$ and $g_b = 1$ near b. If $\{a\} \times T$ is in ∂M_0, then $f_a > 0$, $f_a' > 0$ et $g_a = 1$ near a (changes in signs are due to the orientation convention of the boundary).

By [14, Théorème 4.2], as contact structures $\xi_n = \ker(\alpha_n)$ are universally tight on each components, (M, ξ_n) is universally tight for all $n \in \mathbb{N}^*$. In addition, as our construction correspond to the construction in [15, 4], by Theorem [15, 4.5], there exists infinitely many non-isomorphic ξ_n.

7.3. **Growth rate of contact homology.**

Lemma 7.10. *For all adapted almost complex structures on $\mathbb{R} \times M$, there is no holomorphic cylinder $u = (a, f) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ asymptotic to two Reeb periodic orbits contained in different connected components of $M \setminus (\bigcup_{k=1}^N T_k \times [a, b])$.*

Proof. We prove this result by contradiction. If such a u exists then there exists $k \in 1 \dots N$ such that $f(\mathbb{R} \times S^1) \cap T_k \times \{x\} \neq \emptyset$ for all $x \in [a, b]$. By Lemmas 2.1 and 2.2, there exists x_0 and x_1 in $[a, b]$ such that

(1) $R_{\alpha_n}(x_0) = -R_{\alpha_n}(x_1)$;
(2) $\mathrm{d}u(s,t) \neq 0$ and $\frac{\partial}{\partial \tau} \notin \mathrm{im}(\mathrm{d}u(s,t))$, for all
$$(s,t) \in C = u^{-1}\left(u(\mathbb{R} \times S^1) \cap (\mathbb{R} \times \{x_0, x_1\} \times T)\right).$$

By Lemma 5.3, C is a finite union of smooth circles homotopic to $\{*\} \times S^1$. Cut $\mathbb{R} \times S^1$ along these circles and choose a connected component Σ such that $f(\Sigma) \cap \{x\} \times T \neq \emptyset$ for all $x \in [x_0, x_1]$. Then $\partial f(\Sigma)$ is a union of two homotopic circles: one in $\{x_0\} \times T$ and one in $\{x_1\} \times T$. By positivity of intersection the Reeb vector field is positively transverse to these circles in $\{x_0\} \times T$ and $\{x_1\} \times T$. This leads to a contradiction as $R(x_0) = -R(x_1)$. □

Let Λ_0 be the set of primitive free homotopy classes that correspond to periodic orbits in M_0 and do not represent a homotopy class in a torus $T_k, k = 1 \dots N$. All the Reeb periodic orbits with homotopy class in Λ_0 are non-degenerate. As there is no contractible periodic orbits, the associated partial contact homology is well defined.

There exists $C > 0$ such that all periodic orbits in M_0 associated to a k-periodic point of the first return map h_1 have a period smaller that kC.

Lemma 7.11. *For all $a \in \Lambda_0$, $\dim(HC_*^a(V, \alpha)) \geq 1$. In addition, if a is associated to k-periodic points, for all $L > kC$ the map $HC_{\leq L}^a(V, \alpha) \to HC_*^a(V, \alpha)$ has a rank greater than 1.*

Proof. Choose $a \in \Lambda_0$. Write $C_*^a = C_0 \oplus C_1$ where C_0 by periodic orbits in M_0 homotopic to a and C_1 is generated by periodic orbits in $M \setminus M_0$ homotopic to a. By Lemma 7.10, the differential is written
$$\begin{pmatrix} \partial_a & 0 \\ 0 & * \end{pmatrix}.$$
We prove that $\dim\left(\ker(\partial_a)/\mathrm{im}(\partial_a)\right) \geq 1$. Write $C_0 = E \oplus O$ where E is generated by even periodic orbits and O by odd periodic orbits (as a is primitive all the periodic orbits are good). Then
$$\partial_a = \begin{pmatrix} 0 & \partial_O \\ \partial_E & 0 \end{pmatrix}$$
and
$$\ker(\partial_a)/\mathrm{im}(\partial_a) = \ker(\partial_E)/\mathrm{im}(\partial_O) \oplus \ker(\partial_O)/\mathrm{im}(\partial_E)$$
Hence, $\dim\left(\ker(\partial_a)/\mathrm{im}(\partial_a)\right) = \{0\}$ if and only if $\dim(\ker(\partial_E)) = \dim(\mathrm{im}(\partial_O))$ and $\dim(\ker(\partial_O)) = \dim(\mathrm{im}(\partial_E))$.

By Section 7.1, there exist a branched cover $\hat{S}$ of S and a pseudo-Anosov $\hat{\psi}$ such that the lift $\hat{h}_1$ of h_1 is homotopic to $\hat{\psi}$. Let n denote the Nielsen class associated to the periodic orbits in M_0 homotopic to a and k denote the order of the associated periodic points. Let $\hat{n}$ be a Nielsen class of $\hat{h}_1$ containing a lift of a point in n. As n does not contain points in ∂S, all periodic points in $\hat{n}$ are non-degenerate and there exists s such that $\hat{n}$ contains exactly s lifts of each points in n. By Theorems 7.3 and 7.4, $\Lambda_{\hat{h}_1^k}(\hat{n}) = \Lambda_{\hat{\psi}^k}(\hat{n}) \neq 0$. A periodic point of h_1 is even if and only if the associated Reeb orbit is even. Therefore $\Lambda_{\hat{h}_1^k}(\hat{n}) = s\dim(E) - s\dim(O)$ and $\dim(\ker(\partial_O)) + \dim(\operatorname{im}(\partial_O)) \neq \dim(\ker(\partial_E)) + \dim(\operatorname{im}(\partial_E))$. Hence, $\dim(\ker(\partial_a)/\operatorname{im}(\partial_a)) > 0$.

For all $L > kC$, write $C^a_{\leq L} = C_0 \oplus C_{\leq L}$. The differential is written

$$\begin{pmatrix} \partial_a & 0 \\ 0 & * \end{pmatrix}.$$

Thus, $\dim(\ker(\partial_a)/\operatorname{im}(\partial_a)) \geq 1$ and $HC^a_{\leq L}(V,\alpha) \to HC^a_*(V,\alpha)$ is injective. □

Proof of Theorem 1.3. It remains to prove that the growth rate of $HC^{\Lambda_0}_*(V,\alpha_n)$ is exponential. By Theorems 7.1, 7.3, 7.4 and 7.5, the number of Nielsen classes associated to periodic points of the first return map h_1 grows exponentially. As the number of homotopy classes in tori $T_k, k = 1 \dots N$ exhibits a quadratic growth (Lemma 6.4) and by Lemma 7.11, the growth rate of partial cylindrical homology is exponential. □

Proof of Theorem 1.2. By invariance of the growth rate of partial contact homology (Proposition 4.10), the growth rate of the number of Reeb periodic orbits is exponential if cylindrical contact homology is well defined (Remark 4.6), i.e. if the contact form is non-degenerate and hypertight.

Under Hypothesis H, let α^p_n be a non-degenerate contact form such that $\xi_n = \ker(\alpha^p_n)$. By Theorem 7.7 and Lemma 6.7 there exists an hypertight non-degenerate contact form α'_n of ξ_n. By Theorem 1.3 the growth rate of cylindrical contact homology is exponential. Consider the map $A_*(V,\alpha^p_n,J) \to A_*(V,\alpha'_n,J')$ given by Theorem 2.4 and the pull back augmentation induced by the trivial augmentation on $A_*(V,\alpha'_n,J')$ (Proposition 2.11). By invariance of the growth rate of linearized contact homology (Proposition 4.11), $N_L(\alpha^p_n)$ exhibits an exponential growth with the period. □

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